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Electrolytic Reduction of Menthone

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As to the electrolytic reduction of the hydroaromatic compounds containing the ketonic group much has been studied. Tafel and Schmitz[1] obtained borneol with the 45% current yield by electrolytically reducing camphor in a sulphuric acid solution, and Bredt[2] succeeded in procuring two isomeric borneolcarboxylic acids by the reduction of camphorcarboxylic acid in an alkaline solution. According to these investigations and others the behaviour of the hydroaromatic ketones toward electrolytic reduction is exactly analogous to those of the ordinary ketones, the ketonic group being transformed into the secondary alcoholic group. Sometimes reduction progresses still further so as to change the ketonic group into the hydrocarbon group, as in the case of methylgranatonine[3] from which methylgranatanine is produced as the main reduction product.

The authors of this paper have undertaken to investigate the electrolytic reduction of menthone which if it smoothly takes place, might be expected to have some technical importance for the utilization of the menthone found in peppermint oil.

According to our experiments the electrolytic reduction of menthone was found to take place only in an acid (sulphuric or hydrochloric) solution; in an alkaline solution (20 c.c. of 20% sodium acctate, 45 c.c. of 94% alcohol and 5 grams menthone) no appreciable reaction could be observed. As the cathode material mercury, amalga-

1 Zs. Elektrochem. 8, 288 (1902).

2 Ann. d. Chem. 348, 200 (1906); 366, 1 (1909).

3 A. Picicinini: Gaz. chimi. ital. 32, I, 260 (1902); Chem. Centralbl., 1902, I, 1234.

mated lead, amalgamated tin, lead, tin and nickel were used of which amalgamated lead or tin gave the best results.

The reduction was observed to take place most easily with the mercury cathode which, however, is not suitable for the preparation of menthol, as it greatly increases the formation of an oily substance (see below) lowering the yield of menthol proportionally. With the nickel electrode we could perceive no sign of reduction in the acid solution consisting of 30 c.c. of 75% sulphuric acid, 50 c.c. of 94% alcohol and 5 grams of menthone.

The current efficiency in the above reduction is in no case satisfactory, the yield being not more than 25%.

In all experiments we have always obtained an oily substance which had gradually accumulated on the surface of the cathode solution. The oil boils at about 178–181° and smells rather like a terpene, having no resemblance either to menthone or to menthol in its odour.

We had thought at first that it would be menthene which might have been produced from menthol by the dehydrating action of sulphuric acid. The results of our further investigation, however, were all found contrary to this assumption. First, on passing an electric current into an alcoholic sulphuric acid solution of menthol exactly in the same condition in which menthone was reduced, no oily substance could be obtained. Secondly, on conducting the reduction experiments with sulphuric acid of different dilutions we found the production of the oil the less the more concentrated the sulphuric acid—a fact which causes us to reject the theory that the dehydrating action of sulphuric acid was the cause of the oil formation.

If we admit that the oily substance is not really menthene, the only substance we can now conjecture it to be is menthane, and this view, is supported by the results of its elementary analysis which are fairly well in accord with the composition of menthane, that is $C_{10}H_{20}$.

Moreover, it was found that after 9 or 19 hours' electrolysis with the mercury cathode the amount of hydrogen used up is far more than that required for the complete conversion of total menthone to menthol. Therefore the reduction of menthone to menthol is evidently accompanied by some secondary reaction which leads to the formation of a substance more highly reduced than menthol. That the oily substance is really the highly reduced product of menthone is further shown by the results of the following two experiments. In the first experiment the reduction

efficiency, that is, the ratio of hydrogen really used up to that theoretically required for the complete reduction of menthone to menthol was 1.3, and about 2.2 grams of oil were obtained; in the second experiment the reduction efficiency was 1.4 producing about 3.2 grams of oil. We see therefore the increase in the quantity of oil is parallel with that of the hydrogen consumed.

Experiment 1.

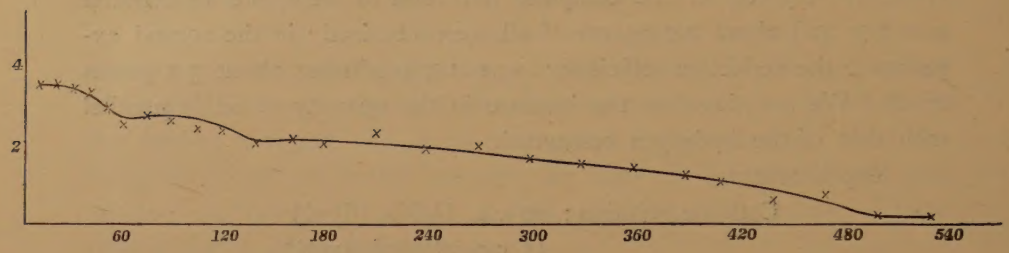
Cathode solution: 20 c.c. H_2SO_4 (65%)
45 c.c. alcohol (94%)
5^{gr} menthone
Cathode: Mercury (28.3 sq. cm.)
Anode solution: H_2SO_4 (65%)
Anode: Platinum
Temperature: 10–13°
Current strength: 3 ampere

Oil produced: 2.2^{gr}
Reduction efficiency: 130%

The electrolysis was continued for 9 hours. A detonating gas coulombmeter was inserted in the circuit, and the hydrogen liberated from the bath and the coulombmeter every 2 minutes was observed at intervals of 3, 5, 7, 10, 15, 20 or 30 minutes. Thus knowing the volume of hydrogen in c.c. used for the reduction during every 1 minute period, the total hydrogen used was roughly calculated from the following diagram.

TABLE I.

Time (minutes)	Hydrogen (c.c.)	Time (minutes)	Hydrogen (c.c.)	Time (minutes)	Hydrogen (c.c.)
10	3·5	120	2·3	330	1·5
20	3·5	140	2·0	360	1·4
30	3·4	160	2·1	390	1·2
40	3·3	180	2·0	410	1·0
50	2·9	210	2·2	440	0·6
60	2·5	240	1·9	470	0·7
75	2·7	270	1·9	500	0·2
90	2·6	300	1·6	530	0·2
105	2·4				

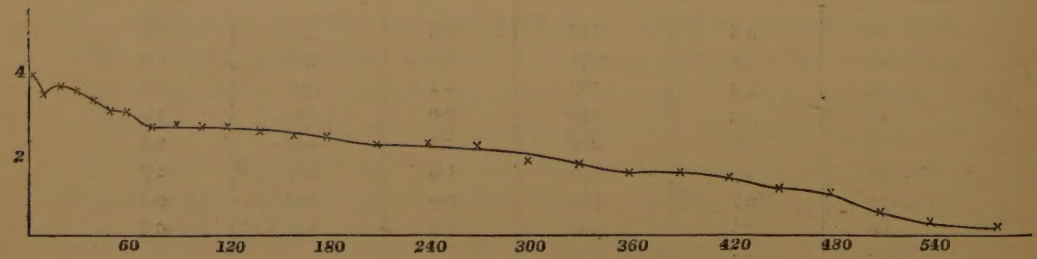


Experiment 2.

In this experiment the more dilute sulphuric acid, that is, 50% solution was employed, other conditions having been maintained exactly the same as in the first experiment. About 3·2 grams of oil were obtained. The following table and diagram show the results.

TABLE II.

Time (minutes)	Hydrogen (c.c.)	Time (minutes)	Hydrogen (c.c.)	Time (minutes)	Hydrogen (c.c.)
3	3·9	105	2·6	330	1·7
10	3·4	120	2·6	360	1·5
20	3·6	140	2·5	390	1·5
30	3·5	160	2·4	420	1·4
40	3·3	180	2·4	450	1·1
50	3·0	210	2·2	480	1·0
60	3·0	240	2·2	510	0·5
75	2·6	270	2·1	540	0·3
90	2·7	300	1.8	570	0·1



We endeavoured to find out the condition in which the reduction progresses solely for the formation of menthol; but all trials failed. The most favourable condition for the production of menthol with the least formation of oil was found as follows: (see table vi)

Cathode solution: 40–45 c.c. H_2SO_4 (75%)
35–40 c.c. alcohol (94%)
5^{gr} menthone
Cathode: Amalgamated lead or tin
Anode solution: Sulphuric acid
Anode: Platinum or lead
Temperature: Below 15°
Current strength: 6–8 ampere/100 sq. cm.
Voltage: 11–13 volt.

Hydrochloric acid may be substituted for H_2SO_4 with nearly the same efficiency (see exp. 12).

Experimental part.

The menthone used in our experiments was prepared from *l*-menthol (m.p. 43°, b.p. 213°, $[\alpha]_D^{25} = -54{\cdot}42$) by oxidising it with a chromic acid mixture (60^{gr} potassium chromate, 50^{gr} concentrated sulphuric acid and 300^{gr} water) according to Beckmann's method[1], and its physical properties were determined as follows: b.p.=207°, $[\alpha]_D^{25} = -25{\cdot}25$, sp. gr.=0·8957.

I. Preliminary experiment.

First to ascertain whether it is possible to reduce menthone electrolytically, the following experiments were performed.

1. The cathode liquid consisting of 25c.c. of 75% sulphuric acid, 50 c.c. of 94% alcohol and 5 grams of menthone was introduced in a beaker, in which a porous cell successively washed with 1% alkali, water, 5% hydrochloric acid and then water was placed as the anode compartment. As the anode liquid 75% sulphuric acid was taken, and a heavy platinum wire in spiral form constituted the anode, while a piece of amalgamated lead about 20 sq. cm. served as the cathode. The beaker was cooled with ice and sodium chloride and a stream of cold water was passed through a spiral tube wound round the anode, to keep the temperature of the bath always below 15°. The electroly-

1 Ann. d. Chem. 250, 325 (1889).

sis was continued for nearly 4 hours with a current of 3 amperes.

Menthol produced by reduction must remain dissolved in the cathode solution, because it is more soluble in alcohol than menthone. Nevertheless, we obtained about 1 gram of an oily substance which, gradually separated out of the cathode liquid and floated on it. On conducting three experiments with the same conditions excepting that the concentration of the acid was lowered to 65, 50 and 25%, we noticed the fact, that the amount of the oily substance increases with the increase of dilution of the acid. When 25% sulphuric acid was used, about 3 grams of oil collected.

2. A hydrochloric acid solution was then tried with the following conditions :

Cathode solution : 25 c.c. HCl (25%)
45 c.c. alcohol (94%)
5 gr menthol

Cathode : Amalgamated lead.
Anode solution : H_2SO_4 (50%)
Anode : Lead
Current strength : 3 ampere
Time : 3 hours

The results showed that reduction proceeds nearly in the same manner as with a sulphuric acid solution and oil is also produced.

The cathode liquid after 3 or 4 hours' electrolysis was treated as is described below.

After separating the oil, the liquid was diluted with much water and extracted with ether. Ether was then evaporated, and the residue was dissolved in alcohol and warmed with hydroxylamine on a water-bath for 15 minutes. The alcohol was then evaporated off, diluted with water and extracted with ether again. The ether extract is now treated repeatedly with sulphuric acid (sp. gr. 1·18) so as to dissolve out menthone oxime. The residue which may be expected to consist mainly of menthol did not crystallize even when cooled with a freezing mixture. This is perhaps due to the presence of some foreign substance, for, according to D. Nishida's investigation[1] the crystallization of menthol is entirely prevented when mixed with 30% of menthone.

[1] Not yet published.

Though the product we obtained does not contain menthone, still it contains the oil above mentioned, the amount of which is, according to our analysis, about 33%. We have therefore sought to separate menthol from the reduction product in the form of phthalic ester. For this purpose the reduction product was mixed with about a half its weight of phthalic anhydride, and after 2 hours' heating at 140–150°, it was subjected to steam distillation, whereupon menthone and the oil were found first to distil over. The phthalic ester thus obtained was then hydrolysed and again distilled with steam. The distillate coming out first crystallized at ordinary temperature, while the later fraction, on being cooled with a freezing mixture.

To make a quantitative determination of the menthol produced 4·5 grams of the reduction product was mixed with 3 grams of acetic anhydride and boiled for 30 minutes, and the acid remaining unchanged was titrated with a standard caustic soda solution. The result showed the presence of 3 grams of menthol.

Menthol isolated as above was analysed with the following results:

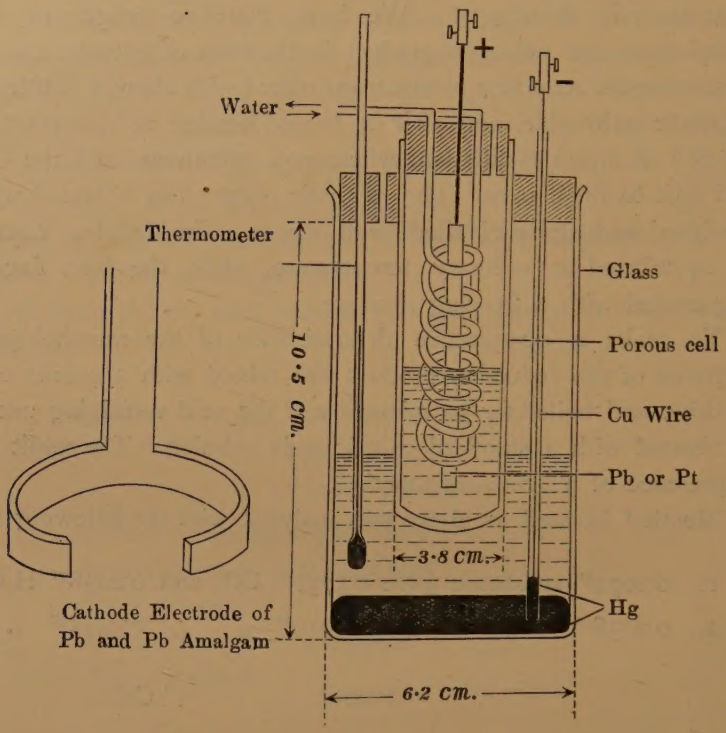
1. 0·2072^{gr} substance gave 0·5757^{gr} CO_2 and 0·2438^{gr} H_2O.
2. 0.2158^{gr} ,, ,, 0·5910^{gr} ,, ,, 0·2451^{gr} ,,

	Found		Cal.
	1	2	
Carbon	75·48	74·70	76·92%
Hydrogen	13·18	12·70	12·82%

II. Quantitative investigation about reduction.

Having now confirmed the fact that menthone can be well reduced to menthol in the acid solution, we have next attempted to know, with what current efficiency the reaction takes place, and how we could obtain the best yield of menthol. For this purpose the electrolysis was carried out in a closed basin as is described before, and hydrogen liberated from a voltameter and from the electrolytic bath was measured. We have carried out more than forty experiments from among which a few typical ones are quoted here.

Fig. 1.



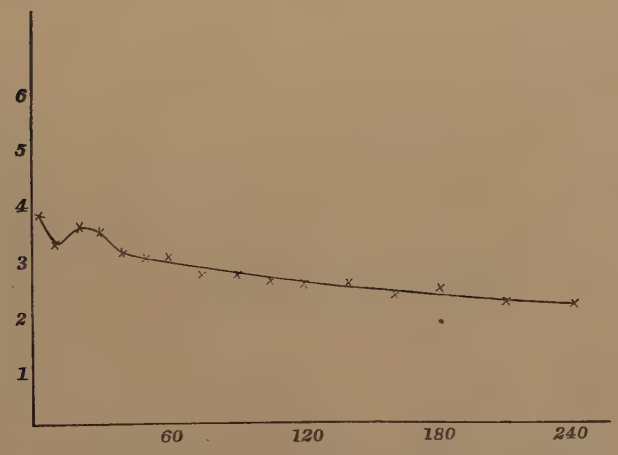
Experiment 3.

Cathode solution: 20 c.c. H_2SO_4 (50%)
45 c.c. alcohol (94%)
5 gr menthone
Cathode: Mercury (28·3 sq. cm.)
Anode solution: 50% H_2SO_4
Anode: Platinum
Temperature: 11–13°
Current strength: 3 ampere
Time: 4 hours

Reduction efficiency: 70% (3 hours)
Current yield: 13·5%
Oil formed: 2·0 gr.

TABLE III.

Time (minutes)	Hydrogen (c.c.)	Time (minutes)	Hydrogen (c.c.)	Time (minutes)	Hydrogen (c.c.)
3	3·9	60	3·0	160	2·35
10	3·35	75	2·7	180	2·4
20	3·6	90	2·7	210	2·2
30	3·5	105	2·6	240	1·15
40	3·25	120	2·55		
50	3·0	140	2·5		



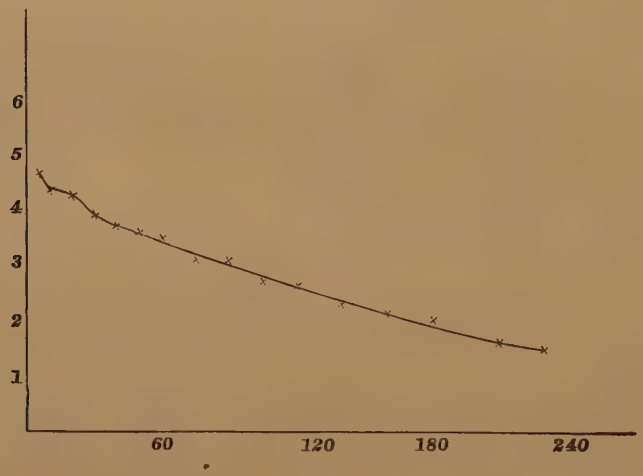
Experiment 4.

20 c.c. of 75% H_2SO_4 was used in the cathode solution, other conditions being the same as in experiment 3.

Reduction efficiency: 75%
Current yield: 15%
Oil formed: 1·0 gr.

TABLE IV.

Time (minutes)	Hydrogen (c.c.)	Time (minutes)	Hydrogen (c.c.)	Time (minutes)	Hydrogen (c.c.)
5	4·6	60	3·5	160	2·1
10	4·3	75	3·1	180	2·0
20	4·2	90	3·1	210	1·6
30	3·9	105	2·7	230	1·5
40	3·7	120	2·6		
50	3·6	140	2·3		



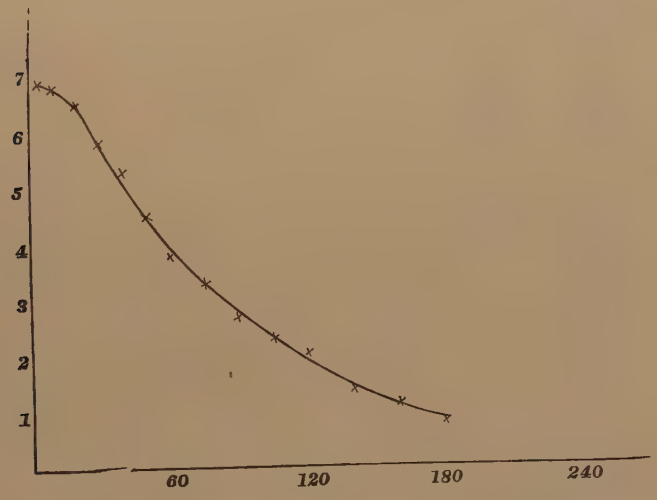
Experiment 5.

Cathode solution: 40 c.c. H_2SO_4 (75%)
40 c.c. alcohol (94%)
5 gr. menthone
Other conditions: As in previous experiments

Reduction efficiency: 80%
Current yield. 23%
Oil formed: 0.5 gr.

TABLE V.

Time (minutes)	Hydrogen (c.c.)	Time (minutes)	Hydrogen (c.c.)	Time (minutes)	Hydrogen (c.c.)
4	6·9	60	3·75	160	1·1
10	6 8	75	3·3	180	0.9
20	6·5	90	2·7		
30	5·8	105	2·3		
40	5·3	120	2·0		
50	4·5	140	1·35		



Experiment 6.

Cathode Solution: 45 c.c. H_2SO_4 (75%)
35 c.c. alcohol (94%)
5 gr. menthone

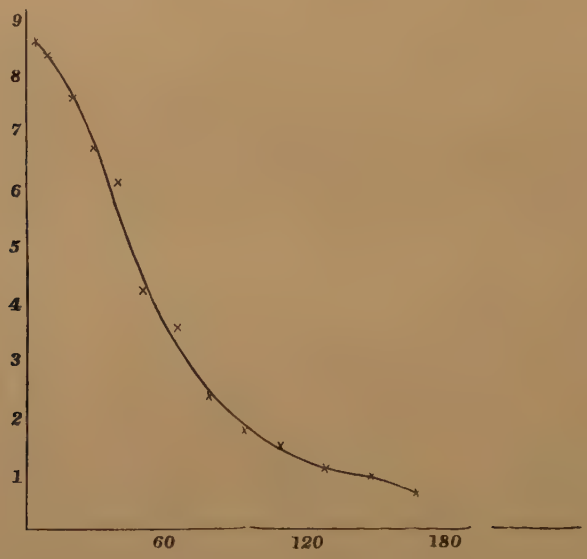
Cathode: Amalgamated lead (28 sq. cm.)

Current strength: 2 ampere

Voltage: 11–13 volt

TABLE VI.

Time (minutes)	Hydrogen (c.c.)	Time (minutes)	Hydrogen (c.c.)	Time (minutes)	Hydrogen (c.c.)
4	8·6	50	4·2	130	1·1
10	8·4	65	3·6	150	1·0
20	7·9	80	2·4	170	0·7
30	6·7	95	1·8		
40	6·1	110	1·5		



Reduction efficiency: 84% (3 hours)
Current yield: 24·3%
Oil formed: 0·2 gr.

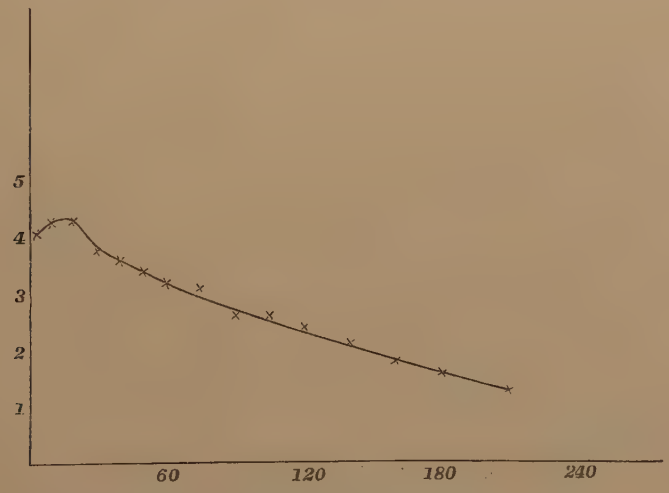
Experiment 7.

Cathode solution: 20 c.c. H_2SO_4 (75%)
45 c.c. alcohol (94%)
5 gr. menthone
Cathode: Amalgamated lead (13 sq. cm.)
Current strength: 2 ampere.

Reduction efficiency: 70% (3 hours)
Current yield: 20%
Oil formed: 0·8 gr.

TABLE VII.

Time (minutes)	Hydrogen (c.c.)	Time (mins.)	Hydrogen (c.c.)	Time (mins.)	Hydrogen (c.c.)
3	4·1	50	3·4	120	2.4
10	4·3	60	3·2	140	2.1
20	4·3	75	3·1	160	1·8
30	3·8	90	2·6	180	1·6
40	3·6	105	2·6	210	1·3



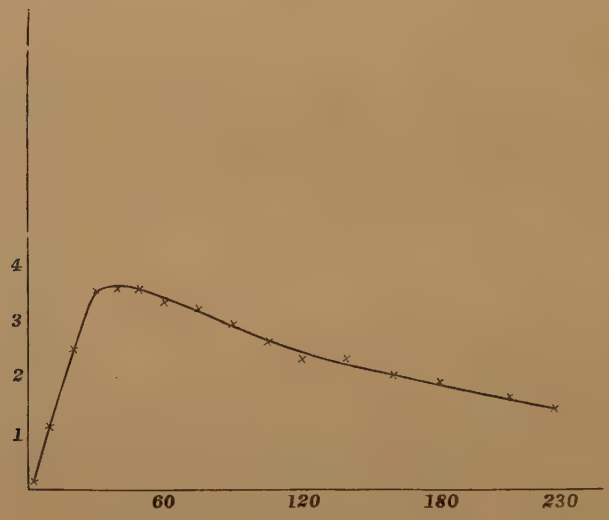
Experiment 8.

Cathode solution: 24 c.c. H_2SO_4 (75%)
54 c.c. alcohol
5 gr. menthone
Cathode: Lead (28 sq. cm.)

Reduction efficiency: 67%
Current yield: 19%
Oil formed: 0·5 gr.

TABLE VIII.

Time (minutes)	Hydrogen (c.c.)	Time (minutes)	Hydrogen (c.c.)	Time (minutes)	Hydrogen (c.c.)
3	0·05	50	3·5	120	2·3
10	1·1	60	3·3	140	2·3
20	2·5	75	3·2	160	2·0
30	3·5	90	2·9	180	1·9
40	3·5	105	2·6	210	1·6
				230	1·4



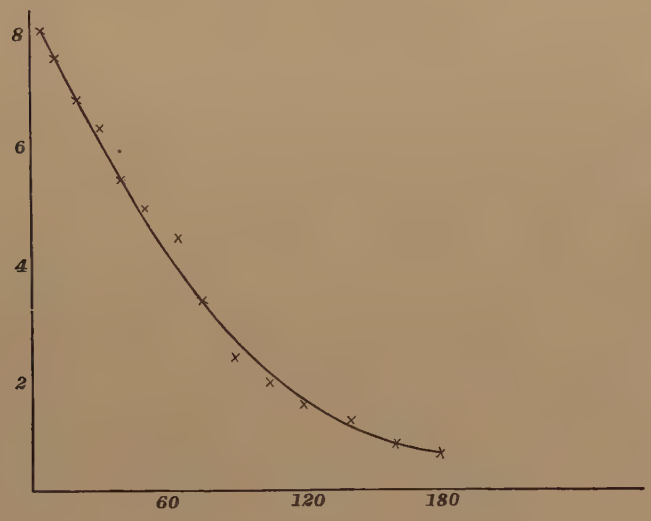
Experiment 9.

Cathode solution: 45 c.c. H_2SO_4 (75%)
35 c.c. alcohol
5 gr. menthone
Cathode: Amalgamated tin (28 sq. cm.)
Temperature: 10–12°
Current strength: 2 ampere
Voltage: 11–12.

Reduction efficiency: 80%
Current yield: 23%
Oil formed: 0·2 gr.

TABLE IX.

Time (minutes)	Hydrogen (c.c.)	Time (minutes)	Hydrogen (c.c.)	Time (minutes)	Hydrogen (c.c.)
5	8·0	40	5·4	105	1·9
10	7·7	50	4·9	120	1·5
20	6·8	60	4·4	140	1·2
30	6·3	75	3·3	160	0·9
		90	2·3	180	0·6



Experiment 10.

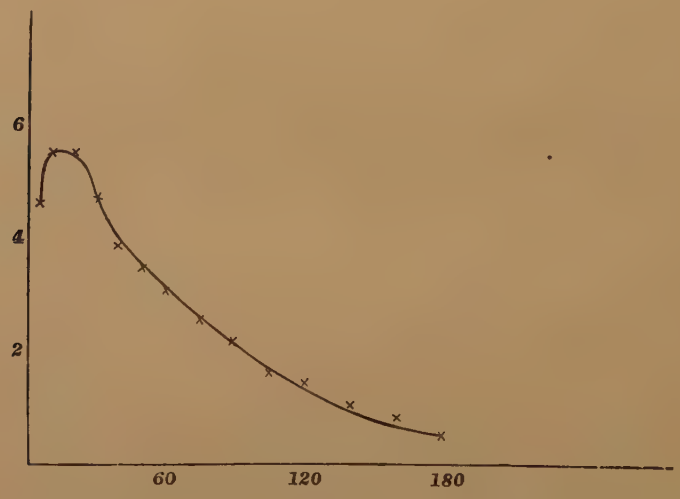
Cathode solution: 25 c.c. H_2SO_4 (75%)
50 c.c. alcohol
5 menthone

Cathode: Amalgamated tin

Reduction efficiency: 62%
Current yield: 18%
Oil formed: 0·3 gr.

TABLE X.

Time (minutes)	Hydrogen (c.c.)	Time (minutes)	Hydrogen (c.c.)	Time (minutes)	Hydrogen (c.c.)
5	4·6	40	3·9	105	1·7
10	5·5	50	3·5	120	1·5
20	5·5	60	3·1	140	1·1
30	4·7	75	2·6	160	0·9
		90	2·2	180	0·6



Experiment 11.

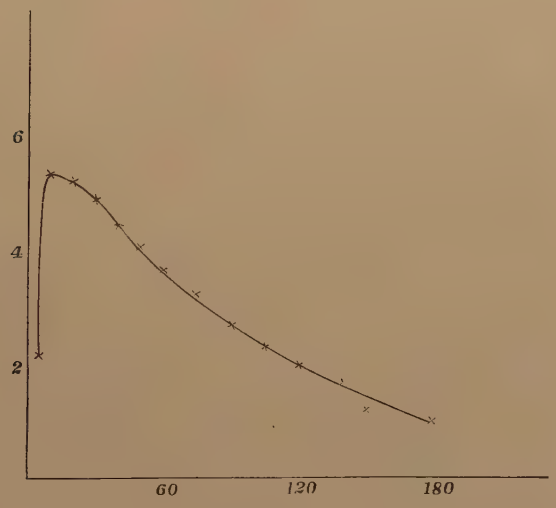
Cathode solution: 35 c.c. H_2SO_4 (75%)
45 c.c. alcohol
5 gr. menthone

Cathode: Tin

Reduction efficiency: 74%
Current yield: 21·5%
Oil formed: 0·4 gr.

TABLE XI.

Time (minutes)	Hydrogen (c.c.)	Time (minutes)	Hydrogen (c.c.)	Time (minutes)	Hydrogen (c.c.)
5	2·2	50	4·1	120	2·0
10	5·4	60	3·7	140	1·7
20	5·3	75	3·3	160	1·2
30	5·0	90	2.7	180	1·0
40	4·5	105	2·3		



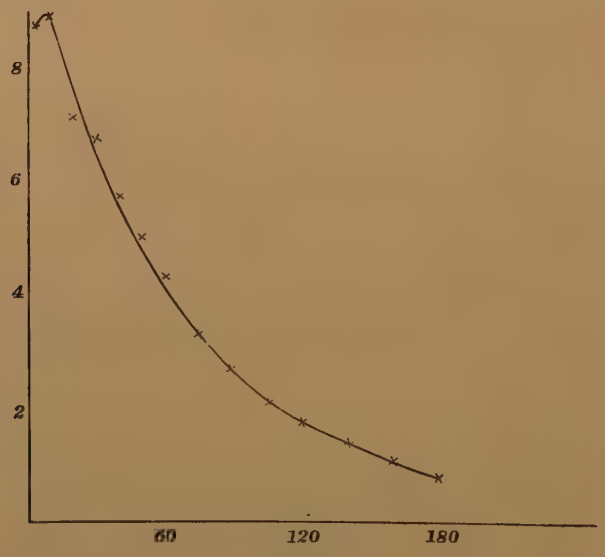
Experiment 12.

Cathode solution: 40 c.c. HCl (40%)
40 c.c. alcohol
5 gr. menthone
Cathode: Amalgamated lead (28 sq. cm.)
Anode solution: Lead
Temperature: 9–11°
Current strength: 2–ampere
Voltage: 10·5–11·5

Reduction efficiency 86%
Current yield: 25%
Oil formed: 0.6 gr.

TABLE XII.

Time (minutes)	Hydrogen (c.c.)	Time (minutes)	Hydrogen (c.c.)	Time (minutes)	Hydrogen (c.c.)
3	8·7	50	5·0	120	1·8
10	8·9	60	4·3	140	1·4
20	7·1	75	3·3	160	1·1
30	6·7	90	2·7	180	0·8
40	5·7	105	2·1		



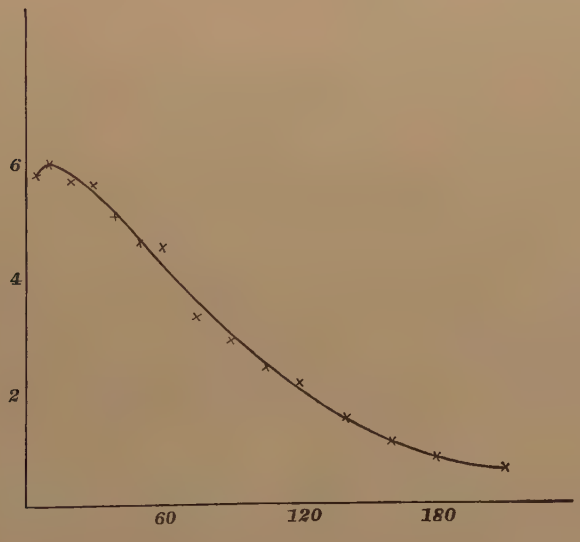
Experiment 13.

The conditions are the same as in the exp. 12 excepting that the current strength is lowered to 1·2 ampere.

Reduction efficiency: 80%
Current yield: 39%
Oil formed: 0·4 gr.

TABLE XIII.

Time (minutes)	Hydrogen (c.c.)	Time (minutes)	Hydrogen (c.c.)	Time (minutes)	Hydrogen (c.c.)
5	5·9	50	4·6	120	2·1
10	6·0	60	4·5	140	1·5
20	5·7	75	3·3	160	1·1
30	5·6	90	2·9	180	0·8
40	5·1	105	2·4	210	0·6



Experiment 14.

Cathode solution : 20 c.c. HCl (25%)
45 c.c. alcohol
5 gr. menthone

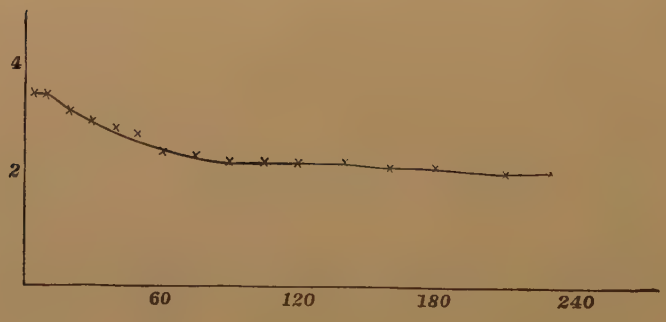
Current strength : 3 ampere

The other conditions being the same as in the previous experiment.

Reduction efficiency : 64%
Current yield : 12·4%
Oil formed : 2·0 gr.

TABLE XIV.

Time (minutes)	Hydrogen (c.c.)	Time (minutes)	Hydrogen (c.c.)	Time (minutes)	Hydrogen (c.c.)
3	3·4	60	2·4	140	2·2
10	3·4	75	2·3	160	2·1
20	3·1	90	2·2	180	2·1
30	2·9	105	2·2	210	2·0
40	2·8	120	2·2	230	2·0
50	2·9				



The Qualitative and Quantitative Determination of Nickel and Cobalt

By

Motooki Matsui and Tadasu Nakazawa

(Received April 21, 1920)

I. Detection of Nickel

Dimethylglyoxime is now generally used as the most sensitive reagent for nickel which forms an insoluble crystalline precipitate on addition of a 1% alcoholic solution of the reagent. As dimethylglyoxime nickel thus formed is soluble both in strong acids and alkalies, it is necessary to make the nickel solution slightly alkaline with ammonia or to add sodium acetate in an excess beforehand.

Now the reaction between dimethylglyoxime and nickel hydroxide (if such reaction actually takes place) might be expected to proceed quantitatively without the use of ammonia or sodium acetate, as no acid is liberated by the reaction thus:

$$2\begin{matrix}CH_3-C=NOH\\ |\\ CH_3-C=NOH\end{matrix} + Ni(OH)_2 = \left(\begin{matrix}CH_3-C=NO\\ |\\ CH_3-C=NOH\end{matrix}\right)_2 Ni + 2H_2O$$

Moreover, if nickel hydroxide directly changes into dimethylglyoxime nickel, the detection of nickel in a mixture of the hydroxides of the iron group metals will be greatly simplified. With these expectations we have experimented in the reaction between dimethylglyoxime and nickel hydroxide.

Nickel hydroxide prepared from the sulphate, after having been repeatedly washed with water, was mixed with a 1% alcoholic solu-

tion of dimethylglyoxime and well shaken, whereupon a reaction instantly took place, and the nickel was completely changed into the glyoxime compound. Ferric hydroxide was observed to have practically no action upon the glyoxime, while cobalt hydroxide partly changes into the glyoxime compound and dissolves. If, therefore, nickel hydroxide is contaminated with the hydroxides of iron and cobalt, especially when small quantities of the former arc mixed with large quantities of other hydroxides, the nickel hydroxide is completely hidden by other hydroxides and escapes detection. This obstacle may, however, be easily overcome in the following way: A mixture is treated with an excess of the dimethylglyoxime solution and boiled for a short time. Water is then added in considerable quantities and the mixture cooled. The solution is now well shaken with ether and left to stand, when dimethylglyoxime nickel floats up with ether, leaving unchanged hydroxides which gradually sink to the bottom of the test-tube.

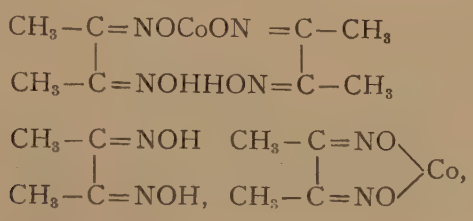
Nickel sulphide also directly changes to dimethylglyoxime nickel, but this action proceeds so slowly, that the quantitative transformation is practically impossible. Glyoxime nickel formed from nickel sulphide may also be separated from the unchanged sulphide by means of ether as described above.

II. Detection of Cobalt

A solution of dimethylglyoxime cobalt is always produced as the filtrate, whenever nickel is separated from cobalt as the glyoxime compound. On endeavouring to find some reaction of the solution which may serve for the detection of cobalt, we found that *when a drop or two of yellow ammonium sulphide is added to it, the solution produces a very intense red colour.* This colour reaction of dimethylglyoxime cobalt is very delicate and does not fail to show the presence of even mere traces of cobalt. High delicacy is attained when a neutral cobalt salt mixed with an excess of a dimethylglyoxime solution is employed. In an ammoniacal solution or in a solution in which an excess of sodium acetate is added, the red colouration is less intense and gradually increases in intensity by standing. It should be remembered that the presence of a large excess of glyoxime is essential, otherwise, cobalt is precipitated as the sulphide. Dimethylglyoxime compound of ferrous iron is also of

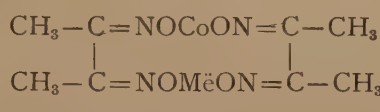
a red colour, so that iron should be previously removed from the solution as ferric hydroxide.

Dimethyl glyoxime cobalt is regarded to have the constitution represented either by

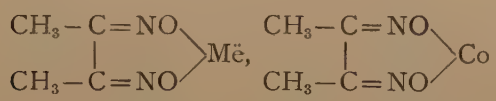


or by

and therefore double compounds, such as



or



may be expected to be derived from it. We have sought some such compound having slight solubility; but so far as our experiments are concerned, all trials ended nagatively. The only compound which has been found to have comparatively slight solubility was lead cobalt compound. It is prepared from a glyoxime cobalt solution on addition of lead acetate as orange yellow crystals. It has the tendency to form a supersaturated solution, so that, from its dilute solution the precipitation takes place only after some time, and is hastened by stirring the liquid with a glass rod.

The results of its analysis are as follows:

	Calc.	Found
Pb	41·65	41·92 %

III. Electrolytic determination of Nickel and Cobalt

The methods usually adopted for quantitative determination of nickel and cobalt in their mixture are those of Fischer[1] and of v.

[1] Pogg. Ann. **72**, 477 (1847).

Liebig[1]. According to the former, cobalt is first precipitated as potassium cobaltinitrite, while by the latter method nickel is separated from cobalt in the form of an oxide by oxidising the mixed complex cyanide solution. Both methods give a satisfactory result when conducted carefully; still the process is somewhat tedious and requires considerable time.

Since the discovery of dimethylglyoxime as the precipitant of nickel, the separation of the two metals has become very simple, and O. Brunck[2] has further confirmed the fact, that the reagent may as well be used for the quantitative analysis of nickel. The most convenient method now employed for the quantitative separation and determination of nickel and cobalt is to determine nickel as the dimethylglyoxime compound on the one hand, and to estimate the total quantities of nickel and cobalt by electrolysis on the other.

Now we have undertaken to see if it is not possible to deposit metallic cobalt quantitatively by electrolysing a dimethyl glyoxime cobalt solution which is obtained as the filtrate whenever nickel is precipitated as the glyoxime compound, and also we have experimented on the electrolytic behaviour of nickel glyoxime.

1. *Electrolysis of Nickel dimethyl glyoxime*

Nickel dimethyl glyoxime was dissolved in a caustic soda solution and electrolysed at the ordinary temperature. When the amount of nickel in the electrolyte was less than 0·05 gram in 100 c.c., an excellent result was obtained; the deposit was of a greyish white colour, contained no trace of carbon, and adhered well to the electrode. Sometimes nickel oxide was observed to be deposited at the anode. In such a case dilute alcohol containing some glycerin (alcohol 45 c.c., glycerin 4 c.c. and water 50 c.c.) was poured into the electrode so as to redissolve it. If the amount of nickel present is more than that above mentioned, or if the temperature of the bath is higher than 50°, the oxide formed at the anode increases and consequently the deposition of metal at the cathode becomes incomplete.

The results of analysis is given in the following table:

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1 Ann. d. Chem. **87**, 128 (1853).

2 Zs angew. Chem. **20**, 834, 1844 (1907).

TABLE I

Electrode	Ni used	Ni found	Electrolyte	Vol. (c.c.)	Volt	Amp	Temp.	Hour
Classen's Basin	0·0203	0·0202	8 gr. NaOH	150	3·2	1·5	18–35°	3·0
,,	0·0406	0·0406	9·6 gr. NaOH	150	3·5	1·5	18–30°	2·5
	0.0203	0·0199	8 gr. NaOH	150	2·7	1·5	60–70°	2·0

2. *Electrolysis of Cobalt dimethyl glyoxime*

The cobalt dimethyl glyoxime solution added with an excess of sodium hydroxide was electrolysed with a result similar to that of the electrolysis of the nickel compound. In this case a much more concentrated solution may be used without fear of the formation of the oxide at the anode; but once formed, it is more difficult to remove. Temperature of the bath should be kept below 50°.

The results are as follows:

TABLE II

Electrode	Co used	Co found	Electrolyte	Vol. (c.c.)	Volt	Amp	Temp	Hour
Classen's Basin	0·0278	0·0276	4 gr. NaOH	100	3·2	0·8	Below 30°	3·0
,,	0·0556	0·0555	5·6 gr. NaOH	100	3·0	0·8	,,	3·2
,,	0·0218	0·0136	4·8 gr. NaOH Dimethyl glyoxime not in excess	100	2.5	0·8	60–70°	2·5

By using ammonia instead of sodium hydroxide, the formation of oxide at the anode can, to some extent, be avoided, the results being shown in table IV.

3. *Separation of Nickel and Cobalt by Electrolysis of their Dimethyl glyoxime compounds*

To a mixed solution of nickel and cobalt sodium acetate was added in an excess, and from this solution nickel was precipitated as the dimethyl glyoxime compound. The precipitate was filtered, and

after having been well washed warmed on a waterbath with a suitable amount of sodium hydroxide. The alkaline solution of dimethyl glyoxime nickel thus obtained was electrolysed. The filtrate containing cobalt was added, with sodium hydroxide, or with ammonia and ammonium sulphate, and was subjected to electrolysis.

The results are shown in the following table:

TABLE III

Electrode	Metal used	Metal found	Electrolyte	Vol. (c.c.)	Volt	Amp	Temp.	Hour
Platinum Net	Ni, 0·0450	0·0448	8 gr. NaOH	100	2·5	1·0	Room temp.	2·5
	Co, 0·0015	0·0014	3·2 gr. „		3·2			1·0
Classen's Basin	Ni, 0·0360	0·0361	6·4 gr. NaOH	100	2·5	1·0	„	„
	Co, 0·0015	0·0014	2·4 gr. NaOH		3·0			
„	Ni, 0·0153	0·0154	4·8 gr. NaOH	100	2·3	1·0	„	1·5
	Co, 0·0015	0·0014	2·0 gr. NaOH		2·8			1·0

TABLE IV

Electrode	Metal used	Metal found	Electrolyte	Vol. (c.c.)	Volt	Amp	Temp	Hour
Net	Ni, 0·0542	0·0540	8 gr. NaOH	100	2·5	1·0	Room temp.	2·5
	Co, 0·0306	0·0305	60 c.c. NH_4OH + 2·6 gr. $(NH_4)_2SO_4$		3·6			2·3
„	Ni, 0·0067	0·0065	4·8 gr. NaOH	100	3·0	0·8	„	1·7
	Co, 0·1836	0·1835	40 c.c. NH_4OH + 5 gr. $(NH_4)_2SO_4$		4·1			3·5
„	Ni, 0·0102	0·0100	4·8 gr. NaOH	100	3·2	1·0	„	2·5
	Co, 0.1836	0·1834	50 c.c. NH_4OH + 2 gr. $(NH_4)_2SO_4$		3·8			3·3

4. Rapid Precipitation

By agitating the electrolyte, the deposition of metal is of course accelerated, but as it is necessary to regulate the current density so as to keep the temperature of the bath below 50° no remarkable acceleration could thus be affected.

TABLE V

Metal	Electrolyte	Vol. (c.c.)	Volt	Amp	Temp	Hour	Deposit
Ni=0·0241	0·1$^{gr.}$ glyoxime 4·8$^{gr.}$ NaOH	100	4	4·5	Below 4·5	0·8	0·0239
Co=0·0612	0·2$^{gr.}$ glyoxime 4·8$^{gr.}$ NaOH	,,	,,	2	,,	1·2	0·0610

5. Conclusion

As the conclusion of the above experiment the following methods of the electrolysis of dimethyl glyoxime nickel and cobalt are deduced.

TABLE VI

Metal	Electrolyte	Vol. (c.c.)	Volt	Amp	Temp.	Hour
Ni or Co 0·01–0·05	0·2$^{gr.}$ dimethyl glyoxime 4·8$^{gr.}$ NaOH	100	2·5–3·0	1·0	Below 50°	1–3

TABLE VII (Rapid method)

Metal	Electrolyte	Vol. (c.c.)	Volt	Amp	Temp.	Minutes	No. of rotation
Ni 0·02–0·05	0·2$^{gr.}$ dimethyl glyoxime 5·5$^{gr.}$ NaOH	100	3·8–4·0	3·5	Below 50°	50	800
Co ,,	0·2$^{gr.}$ dimethyl glyoxime 6·5$^{gr.}$ NaOH	,,	4·0–4·5	,,	,,	,,	,,

Oriented Circles in Non-Euclidean Space,

By

Teikichi Nishiuchi and Hidetoshi Kashiwagi.

(Received, April 26, 1920)

CHAPTER I

§ 1

Coordinates of an Oriented Circle in Non-Euclidean Space.

Circles in euclidean space have been studied already by Laguerre, Koenigs, Stephanos, Cossert, Castelnuovo and Coolidge. But the study of circles in non-euclidean space has not been taken up by anybody. We will discuss the oriented circles by a method analogous to that of R. Bricard[1] in his paper 'Géométrie des feuillets' which is the generalization of line-geometry and that of Study.[2] And we shall consider a circle in non-euclidean space as the intersection of a sphere and a plane. Thus any sphere and any plane can define a circle, and without any loss of generality, we may consider a circle as the intersection of a sphere and a plane passing through the center of the sphere.

According to the definition, a circle in non-euclidean space may be represented by the equations of the form:

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1 See B. Bricard, Comptes Rendus, Tome. 150, 1910, p. 1586. See also Nouvelles Annales de Mathématiques, series. 4, Tome. x.

2 See Study, Geometrie der Dynamen.

$$(a,x)^2=\cos^2\frac{r}{k}(x,x),$$

$$(b,x)=0$$

with the condition

$$(a,a)=1,\quad (a,b)=0,\quad (b,b)=1,$$

where (a) is the coordinates of the center of the sphere, (b) that of the plane of the circle, r the radius of the sphere and k the measure of curvature of the space. The center of the circle coincides with the center of the sphere. We shall call the absolute pole of the plane of the circle *absolute pole of the circle* and the sphere *main sphere of the circle*. The absolute polar plane of the center of the circle shall be called *central plane of the circle* and the intersection of the central plane of the circle and the plane of the circle *plane-axis of the circle*. The line perpendicular to the plane of the circle passing through the center of the circle shall be called *space-axis of the circle*.

The axial coordinates of the plane-axis are

$$\pi_{ij}=\begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix},$$

and the radical coordinates of the space-axis are

$$p_{ij}=\begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix},$$

and these two lines are evidently absolute polars to each other.

Next, we shall define *a polar sphere* as the sphere which is the envelope of the absolute polar planes of the points on a given sphere. The equation to the polar sphere of a given sphere

$$(a,x)^2=\cos^2\frac{r}{k}(x,x).$$

is

$$(a,x)^2=\sin^2\frac{r}{k}(x,x).$$

And we shall define the intersection of the plane of a circle and the polar sphere of the main sphere as the *polar circle*.

Now, we shall define *the foci of the circle* as the centers of the null-sphere passing through the given circle in analogy to Laguerre's

definition[1] in euclidean space. According to this definition, we have two foci whose coordinates are

$$(a \pm i \sin\frac{r}{k} b), \text{ where } i = \sqrt{-1},$$

and the coordinates of two foci of the polar circle are

$$(a \pm i \cos\frac{r}{k} b), \text{ where } i = \sqrt{-1}.$$

The foci may be considered as the intersecting points of the isotropic lines passing through the circle. We shall define an *oriented circle* as a figure formed by a circle and the two foci of the polar circle in that order, calling one the *first* and the other the *second*. When the rôles of the two foci have been interchanged, the resulting oriented circle shall be called to be the *opposite* of the original one. A circle whose radius $\frac{\pi}{2} k$ is its own opposite.

We shall define the common points of two oriented circles which lie on one of the radical planes of the main spheres

$$(a,x)\cos\frac{r'}{k} - (a',x)\cos\frac{r}{k} = 0$$

their points of intersection, where the equations of the two main spheres are

$$(a,x)^2 = \cos^2\frac{r}{k}(x,x), \quad (a',x)^2 = \cos^2\frac{r'}{k}(x,x).$$

Hence, two oriented circles can not intersect at more than two points. When two oriented circles are coplanar, they will intersect at two points and the line joining these points of intersection is given by the equation

$$(a,x)\cos\frac{r'}{k} - (a',x)\cos\frac{r}{k} = 0,$$

where the equations of the two coplanar circles are

$$(a,x)^2 = \cos^2\frac{r}{k}(x,x), \quad (a',x)^2 = \cos^2\frac{r'}{k}(x,x)$$

1 See Laguerre, 'Sur l'emploi des imaginaires dans la Géométrie de l'espace,' Nouvelles, Annales de Mathématiques, series. 2, vol. XI, 1872.

and cosine of the angles of intersection will be

$$\cos\theta = \frac{(a,a') - \cos\frac{r}{k}\cos\frac{r'}{k}}{\sin\frac{r}{k}\sin\frac{r'}{k}}. \quad (1)$$

Let $(\bar{a}_0, \bar{a}_1, \bar{a}_2, \bar{a}_3)$ and $(\bar{b}_0, \bar{b}_1, \bar{b}_2, \bar{b}_3)$ be Weirstrass' coordinates of the center of a circle and the pole of its plane, and we put

$$a_0 \equiv \bar{a}_0, \quad a_1 \equiv \frac{\bar{a}_1}{k}, \quad a_2 \equiv \frac{\bar{a}_2}{k}, \quad a_3 \equiv \frac{\bar{a}_3}{k};$$

$$b_0 \equiv \bar{b}_0, \quad b_1 \equiv \frac{\bar{b}_1}{k}, \quad b_2 \equiv \frac{\bar{b}_2}{k}, \quad b_3 \equiv \frac{\bar{b}_3}{k},$$

then

$$(a,a) = 1, \quad (a,b) = 0, \quad (b,b) = 1.$$

We shall call

$$a_0, a_1, a_2, a_3, i\cos\frac{r}{k}b_0, i\cos\frac{r}{k}b_1, i\cos\frac{r}{k}b_2, i\cos\frac{r}{k}b_3$$

the coordinates of an oriented circle. Now, we adopt the following notation:

$$a_0, a_1, a_2, a_3, i\cos\frac{r}{k}b_0, i\cos\frac{r}{k}b_1, i\cos\frac{r}{k}b_2, i\cos\frac{r}{k}b_3$$

$$\equiv p_0, p_1, p_2, p_3, q_0, q_1, q_2, q_3$$

$$\equiv \begin{pmatrix} p \\ q \end{pmatrix}$$

where we always take $a_0 > 0$, $b_0 > 0$, if $a_0 = 0$ (or $b_0 = 0$) then $a_1 > 0$ (or $b_1 > 0$) and so on, and

$$\sqrt{(a,a)} = 1, \qquad \sqrt{(b,b)} = 1.$$

It is evident that all the determinants of the matrix

$$\left\| \begin{matrix} p_0 & p_1 & p_2 & p_3 \\ q_0 & q_1 & q_2 & q_3 \end{matrix} \right\|$$

are not equal to zero, for if all of them vanish, then

$$a_i = b_i \qquad (i = 0, 1, 2, 3)$$

which is absurd.

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1 See Coolidge, Non-Euclidean Geometry, p. 137.

We see that there exists a relation

$$(p, q) \equiv p_0 q_0 + p_1 q_1 + p_2 q_2 + p_3 q_3 = 0,$$

which is analogous to Plücker's identity of the line-coordinates. So we shall call the coordinates $\begin{pmatrix} p \\ q \end{pmatrix}$ *Plücker's coordinates of the oriented circle.*

From the definition of the polar circle, it is evident that an oriented circle and its polar circle are concentric and coplanar, but their radii differ by $-\frac{\pi}{2}k$. Hence, Plücker's coordinates of the polar circle of an oriented circle whose coordinatas $\begin{pmatrix} p \\ q \end{pmatrix}$ are

$$a_0, a_1, a_2, a_3, i\sin\frac{r}{k}b_0, i\sin\frac{r}{k}b_1, i\sin\frac{r}{k}b_2, i\sin\frac{r}{k}b_3$$

and we shall denote it by the notation

$$\bar{p}_0, \bar{p}_1, \bar{p}_2, \bar{p}_3, \bar{q}_0, \bar{q}_1, \bar{q}_2, \bar{q}_3 \equiv \begin{pmatrix} \bar{p} \\ \bar{q} \end{pmatrix}.$$

We can easily see that there exists a following relation between Plücker's coordinates of an oriented circle and its polar circles:

$$(p, p) + (q, q) + (\bar{p}, \bar{p}) + (\bar{q}, \bar{q}) = 1.$$

A circle whose radius is zero is called a null-circle. The coordinates of a null-circle satisfy the following condition:

$$(p, p) + (q, q) = 0.$$

Again we put

$$\begin{aligned}
\mathfrak{X}_0 &\equiv p_0 + q_0, \\
\mathfrak{X}_1 &\equiv p_1 + q_1, \\
\mathfrak{X}_2 &\equiv p_2 + q_2, \\
\mathfrak{X}_3 &\equiv p_3 + q_3, \\
{}_0\mathfrak{X} &\equiv i(p_0 - q_0), \\
{}_1\mathfrak{X} &\equiv i(p_1 - q_1), \\
{}_2\mathfrak{X} &\equiv i(p_2 - q_2), \\
{}_3\mathfrak{X} &\equiv i(p_3 - q_3), \quad \text{where } i = \sqrt{-1}.
\end{aligned}$$

Then we notice that

$$(\mathfrak{X}_i) = \mathfrak{X}_0 : \mathfrak{X}_1 : \mathfrak{X}_2 : \mathfrak{X}_3$$

and

$$({}_i\mathfrak{X}) = {}_0\mathfrak{X} : {}_1\mathfrak{X} : {}_2\mathfrak{X} : {}_3\mathfrak{X}$$

are respectively the coordinates of the foci of the polar circle of the given oriented circle.

The oriented circle may be represented by

$$\mathfrak{X}_0, \mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3, {}_0\mathfrak{X}, {}_1\mathfrak{X}, {}_2\mathfrak{X}, {}_3\mathfrak{X} \equiv \begin{pmatrix} \mathfrak{X}_j \\ {}_j\mathfrak{X} \end{pmatrix}$$

and Plücker's identity becomes as follows:

$$(\mathfrak{X}, \mathfrak{X}) = (\mathfrak{X}_j, \mathfrak{X}_j) + ({}_j\mathfrak{X}, {}_j\mathfrak{X}) = 0,$$

which is analogous to the Klein identity of the line-coordinates. Hence, we shall call $\begin{pmatrix} \mathfrak{X}_j \\ {}_j\mathfrak{X} \end{pmatrix}$ *Klein's coordinates of the oriented circle.* So the coordinates of the polar circle of the given oriented circle are represented by

$$\begin{pmatrix} \overline{\mathfrak{X}}_j \\ {}_j\overline{\mathfrak{X}} \end{pmatrix}$$

where

$$\overline{\mathfrak{X}}_j \equiv \bar{p}_j + \bar{q}_j,$$
$${}_j\overline{\mathfrak{X}} \equiv i(\bar{p}_j - \bar{q}_j), \qquad (i=\sqrt{-1},\ j=0, 1, 2, 3).$$

There exist the following relations between $(\mathfrak{X}_j)$, $({}_j\mathfrak{X})$ and $(\overline{\mathfrak{X}}_j)$, $({}_j\overline{\mathfrak{X}})$:

$$(\mathfrak{X}_j, \mathfrak{X}_j) + (\overline{\mathfrak{X}}_j, \overline{\mathfrak{X}}_j) = 1,$$
$$({}_j\mathfrak{X}, {}_j\mathfrak{X}) + ({}_j\overline{\mathfrak{X}}, {}_j\overline{\mathfrak{X}}) = -1.$$

If the given oriented circle is null, then we have

$$(\mathfrak{X}_j, \mathfrak{X}_j) = ({}_j\mathfrak{X}, {}_j\mathfrak{X}) = 0.$$

§ 2

Relations between two Oriented Circles.

Suppose two circles[1] with Plücker's coordinates $\begin{pmatrix} p \\ q \end{pmatrix}$ and $\begin{pmatrix} p' \\ q' \end{pmatrix}$ are given.

If $$p_i = p_i' \qquad (i=0, 1, 2, 3),$$
then they are concentric.

If $$q_i = q_i' \qquad (i=0, 1, 2, 3),$$
then they are coplanar, or copolar.

If
$$(p, p') = 0,$$
then their centers are orthogonal, or the central planes cut orthogonally to each other.

If
$$(p, q') = 0,$$

[1] For the sake of brevity, we shall call, here in after, an oriented circle a circle.

then the center of one circle lies on the plane of the other, or the center of one circle and the pole of the other circle are mutually orthogonal.

If $$(q, q')=0,$$

then the planes of the circles cut orthogonally, or the poles of the circles are mutually orthogonal, when the radius of both of the circles is not equal to $\frac{\pi}{2}k$.

If $$(q, q)=(q', q'),$$

then the two circles are equiradii.

If $$(\bar{p}, \bar{q}')+(p', \bar{q})=0,$$

then we have

$$\frac{\sin\frac{r}{k}}{\cos\left(\frac{\overline{ab'}}{k}\right)} : \frac{\sin\frac{r'}{k}}{\cos\left(\frac{\overline{a'b}}{k}\right)}=-1,$$

or

$$\frac{\sin\frac{r}{k}}{\sin\frac{d}{k}} : \frac{\sin\frac{r'}{k}}{\sin\frac{d'}{k}}=-1,$$

where

$$\cos\left(\frac{\overline{ab'}}{k}\right)=\sin\frac{d}{k},\quad \cos\left(\frac{\overline{a'b}}{k}\right)=\sin\frac{d'}{k}.$$

This relation is elegant, this means that one set of the four points on the line of centers cut by the two main spheres of the circles and two other spheres of which each is described with a center, the center of one circle, and touching the plane of the other forms a harmonic group.

If $$(p, q')+(p', q)=0,$$

then the angles formed by the radical plane of the main spheres

$$(a, x)\cos\frac{r'}{k}-(a', x)\cos\frac{r}{k}=0$$

and the planes of the circles are equal.

If two non-coplanar circles whose radii are not equal to $\frac{\pi}{2}k$ are cospherical, then, first, their space-axes should cut each other. Let the coordinates of the point of the intersection of their space-axes be $(a+\lambda b)$ or $(a'+\lambda' b')$, then

$$\frac{a_0+\lambda b_0}{a_0'+\lambda' b_0'}=\frac{a_1+\lambda b_1}{a_1'+\lambda' b_1'}=\frac{a_2+\lambda b_2}{a_2'+\lambda' b_2'}=\frac{a_3+\lambda b_3}{a_3'+\lambda' b_3'}$$

and

$$\frac{\cos\frac{r}{k}}{\sqrt{1+\lambda^2}}=\frac{\cos\frac{r}{k}}{\sqrt{1+\lambda'^2}}.$$

Hence, eliminating λ and λ' from the above equations, we get the following conditions that the two non-coplanar circles are cospherical

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ a_0' & a_1' & a_2' & a_3' \\ b_0' & b_1' & b_2' & b_3' \end{vmatrix}=0,$$

and all the determinants of the matrix

$$\begin{Vmatrix} a_0\cos\frac{r'}{k}-a_0'\cos\frac{r}{k} & a_1\cos\frac{r'}{k}-a_1'\cos\frac{r}{k} & \cdots\cdots & a_3\cos\frac{r'}{k}-a_3'\cos\frac{r}{k} \\ b_0 & b_1 & \cdots\cdots & b_3 \\ b_0' & b_1' & \cdots\cdots & b_3' \end{Vmatrix}$$

are equal to zero. Or in Plücker's coordinates

$$\begin{vmatrix} p & q & p' & q' \end{vmatrix}=0$$

and all the determinants of the matrix

$$\begin{Vmatrix} p_0\sqrt{(q',q')}-p_0'\sqrt{(q,q)} & p_1\sqrt{(q',q')}-p_1'\sqrt{(q,q)} & \cdots\cdots & p_3\sqrt{(q',q')}-p_3'\sqrt{(q,q)} \\ q_0 & q_1 & \cdots\cdots & q_3 \\ q_0' & q_1' & \cdots\cdots & q_3' \end{Vmatrix}$$

are equal to zero.

If two circles intersect at one point only, the point of the intersection should lie on the radical plane of their main sphere

$$(a,x)\cos\frac{r'}{k}-(a',x)\cos\frac{r}{k}=0$$

and the planes of the circles

$$(b, x)=0, \quad (b', x)=0.$$

Hence, when they are not coplaner and cospherical, the coordinates of the point of intersection are given by the ratio of the determinants of the matrix

$$\begin{Vmatrix} p_0\sqrt{(q',q')}-p_0'\sqrt{(q,q)} & p_1\sqrt{(q',q')}-p_1'\sqrt{(q,q)} & \cdots\cdots & p_3\sqrt{(q',q')}-p'_3\sqrt{(q,q)} \\ q_0 & q_1 & \cdots\cdots & q_2 \\ q_0' & q_1' & \cdots\cdots & q_2' \end{Vmatrix}$$

If two circles intersect at two points, then the radical plane of their main spheres

$$(a, x)\cos\frac{r'}{k}-(a', x)\cos\frac{r}{k}=0$$

and the planes of the circles are coaxial, hence all the determinants of the above matrix are equal to zero.

§ 3

On Circles and Their Main Spheres.

Theorem. The necessary and sufficient condition that the two main spheres of two coplanar circles whose coordinates $\begin{pmatrix} p \\ q \end{pmatrix}$ and $\begin{pmatrix} p' \\ q' \end{pmatrix}$ should touch each other is that

$$\frac{(p,p')+(q,q')}{\sqrt{(p,p)+(q,q)}\sqrt{(p',p')+(q',q')}}=\pm 1.$$

Theorem. The necessary and sufficient condition that the two main spheres of two coplanar circles should cut orthogonally is that

$$(p, p')+(q, q')=0.$$

Theorem. If the planes of the two circles are mutually orthogonal, the necessary and sufficient condition that their main spheres should cut orthogonally is that

$$(p, p') + \sqrt{(q, q)}\sqrt{(q', q')} = 0.$$

Theorem. If the centers of two circles lie on the intersection of the planes of the circles and their main spheres cut orthogonally, then the first main sphere passes through the foci of the second circle and the second main sphere passes through the foci of the first.

Theorem. If the two circles are concentric and

$$(q, q')^2 = 1,$$

then the foci of the two circles are mutually orthogonal.

Theorem. If the two circles are concentric and their planes are mutually orthogonal, then the distances between the foci of the two circles are equal to zero.

If

$$\frac{(\bar{p}, \bar{p}') + (\bar{q}, \bar{q}')}{\sqrt{(\bar{p}, \bar{p}) + (\bar{q}, \bar{q})}\sqrt{(\bar{p}', \bar{p}') + (\bar{q}', \bar{q}')}} = 1,$$

then

$$\frac{(a, a') - \cos\frac{r}{k}\cos\frac{r'}{k}}{\sin\frac{r}{k}\sin\frac{r'}{k}} = (b, b')$$

or

$$\cos\theta = \cos\varphi,$$

where φ is the angle between the planes of the circles and θ is one of the angles formed by their main spheres. Hence, we have the following theorem.

Theorem. If there exist the following relation between the Plücker coordinates $\begin{pmatrix} p \\ q \end{pmatrix}$ and $\begin{pmatrix} p' \\ q' \end{pmatrix}$ of two circles:

$$\frac{(\bar{p}, \bar{p}') + (\bar{q}, \bar{q}')}{\sqrt{(\bar{p}, \bar{p}) + (\bar{q}, \bar{q})}\sqrt{(\bar{p}', \bar{p}') + (\bar{q}', \bar{q}')}} = 1,$$

then the angle between the two planes of the circles and one of the angles formed by their main spheres are equal.

§ 4

Common Orthogonal Sphere of two Circles.

When a sphere is orthogonal to every sphere through a circle, we shall call the sphere orthogonal to the circle.

Suppose a sphere

$$(ax)^2=\cos^2\frac{R}{k}(a,a)(x,x)$$

be orthogonal to a circle whose Plücker's coordinates are

$$\begin{pmatrix} p \\ q \end{pmatrix}\equiv a_0,\ a_1,\ a_2,\ a_3,\ i\cos\frac{r}{k}b_0,\ i\cos\frac{r}{k}b_1,\ i\cos\frac{r}{k}b_2,\ i\cos\frac{r}{k}b_3,$$

then

$$(a,b)=0$$

and

$$\frac{(a,a)}{\sqrt{(a,a)}}=\cos\frac{R}{k}\cos\frac{r}{k}.$$

Hence, we get the following theorem.

Theorem. The necessary and sufficient conditions that a sphere be orthogonal to a circle are that the center of the sphere should lie on the plane of the circle and the given sphere should cut the main sphere of the circle orthogonally.

Theorem. If a sphere is orthogonal to a circle, then the sphere will pass through the foci of the circle.

Now we will find the common orthogonal sphere of two non-cospherical and non-coplanar circle with Plücker's coordinates $\begin{pmatrix} p \\ q \end{pmatrix}$ and $\begin{pmatrix} p' \\ q' \end{pmatrix}$.

Let the equation of the required sphere be

$$(a,x)^2=\cos^2\frac{R}{k}(a,a)(x,x),$$

then

$$(a,b)=(a,b')=0,$$

$$\frac{(a,a)}{\sqrt{(a,a)}}=\cos\frac{R}{k}\cos\frac{r}{k}$$

and

$$\frac{(a',a)}{\sqrt{(a,a)}}=\cos\frac{R}{k}\cos\frac{r'}{k}.$$

Therefore, the coordinates of the center of the common orthogonal sphere are

$$\rho a_i = \frac{\partial}{\partial t_i} \left| t \; a\cos\frac{r'}{k} - a'\cos\frac{r}{k} \;\; b \;\; b' \right|^{(1)}$$

or

$$\rho' a_i = \frac{\partial}{\partial t_i} \left| t \; p\sqrt{(q', q')} - p'\sqrt{(q, q)} \;\; q \;\; q' \right|$$

where ρ, ρ' are proportional factors and $i=o,\ 1,\ 2,\ 3$. And the radius of the common orthogonal sphere is given by the formula

$$\cos\frac{R}{k} = \frac{(a, a)}{\sqrt{(a, a)}\cos\frac{r}{k}} = \frac{(a', a)}{\sqrt{(a, a)}\cos\frac{r'}{k}}$$

By this discussion, we see that the common orthogonal sphere will pass through the foci of the two circles and the center of the sphere lies on the intersection of the planes of the circles and the length of the tangents from the center of the sphere to the two given circles are equal to the radius of the sphere. If two non-coplanar circles intersect at one point, then their point of intersection will coincide with the center of the common orthogonal sphere and the sphere will be a null-sphere. So the condition that two non-coplanar circles should intersect is that

$$\frac{(a, a)}{\sqrt{(a, a)}\cos\frac{r}{k}} = \frac{(a', a)}{\sqrt{(a, a)}\cos\frac{r'}{k}} = 1,$$

or

---

1 This notation means the coordinates (α) are the ratio of the determinants of the matrix

$$\left\| \begin{matrix} a_o\cos\frac{r'}{k} - a_o'\cos\frac{r}{k} & a_1\cos\frac{r'}{k} - a'_1\cos\frac{r}{k} & a_2\cos\frac{r'}{k} - a'_2\cos\frac{r}{k} & a_3\cos\frac{r'}{k} - a'_3\cos\frac{r}{k} \\ b_o & b_1 & b_2 & b_3 \\ b'_o & b'_1 & b'_2 & b'_3 \end{matrix} \right\|$$

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ a'_0 & a'_1 & a'_2 & a'_3 \\ b'_0 & b'_1 & b'_2 & b'_3 \end{vmatrix}^2$$

$$= \left\| \begin{matrix} a_0\cos\frac{r'}{k} - a'_0\cos\frac{r}{k} & a_1\cos\frac{r'}{k} - a'_1\cos\frac{r}{k} & \cdots\cdots & a_3\cos\frac{r'}{k} - a'_3\cos\frac{r'}{k} \\ b_0 & b_1 & \cdots\cdots & b_2 \\ b'_0 & b'_1 & \cdots\cdots & b'_2 \end{matrix} \right\|^2$$

i. e

$$\begin{vmatrix} 1 & 0 & (a,a') & (a,b') \\ 0 & 1 & (b,a') & (b,b') \\ (a',a) & (a',b) & 1 & 0 \\ (b',a) & (b',b) & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \cos^2\frac{r'}{k} - 2(a,a')\cos\frac{r}{k}\cos\frac{r'}{k} + \cos^2\frac{r}{k} & -(a',b)\cos\frac{r}{k} & (a,b')\cos\frac{r'}{k} \\ -(a',b)\cos\frac{r}{k} & 1 & (b,b') \\ (a,b')\cos\frac{r'}{k} & (b',b) & 1 \end{vmatrix},$$

or

$$\begin{vmatrix} 1 & 0 & (p,p') & (p,q') \\ 0 & 1 & (q,p') & (q,q') \\ (p',p) & (p',q) & 1 & 0 \\ (q',p) & (q',q) & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} (q',q') - 2(p,p')\sqrt{(q,q)}\sqrt{(q',q')} + (q,q) & -(p',q)\sqrt{(q,q)} & (p,q')\sqrt{(q',q')} \\ -(p',q)\sqrt{(q,q)} & 1 & (q,q') \\ (p,q')\sqrt{(q,q')} & (q',q) & 1 \end{vmatrix}.$$

From this condition and that in §2, we see that when two non-coplanar circles intersect at two points they are cospherical.

A system of circles which is orthogonal to a sphere shall be called *a normal system of circles*. The planes of the circles of this system pass through a common point which is the center of the

sphere, and the length of every tangent from this point to the circles of this system are all equal to the radius of the sphere. Moreover the points of contact of these tangents and the foci of the circles lie on the sphere.

§ 5

The Angle Between Two Circles at Their Points of Intersection.

We can define the angle between two circles at their points of intersection by the angle of the tangents to the circles at that point.

Let (y) be the coordinates of a point of intersection of two circles with the coordinates $\begin{pmatrix} p \\ q \end{pmatrix}$ and $\begin{pmatrix} p' \\ q' \end{pmatrix}$. The plane-coordinates of the tangent planes to the main spheres at that point are

$$\lambda u_i=(a,y)\ a_i-\cos^2\frac{r}{k}y_i,$$

$$\lambda' u'_i=(a',y)\ a'_i-\cos^2\frac{r'}{k}y_i, \qquad (i=0,1,2,3)$$

where λ and λ' are proportional factors. Hence, the axial-coordinates of the tangents to the two circles at the point (y) are

$$\pi_{ij}=\begin{vmatrix} u_i & u_j \\ b_i & b_j \end{vmatrix},$$

$$\pi'_{ij}=\begin{vmatrix} u'_i & u'_j \\ b'_i & b'_j \end{vmatrix}.$$

The cosine of the angle formed by them (i. e. the cosine of the angle of the circles at the point of intersection) will be

$$\cos\theta=\frac{\Sigma\pi_{ij}\,\pi'_{ij}}{\sqrt{\Sigma\pi_{ij}^2}\,\sqrt{\Sigma\pi'^2_{ij}}}$$

$$=\frac{\begin{vmatrix} (u,u') & (u,b') \\ (b,u') & (b,b') \end{vmatrix}}{\sqrt{\begin{vmatrix} (u,u) & (u,b) \\ (b,u) & (b,b) \end{vmatrix}}\cdot\sqrt{\begin{vmatrix} (u',u') & (u',b') \\ (b',u') & (b',b') \end{vmatrix}}}.$$

But

$$(a,y)^2=\cos^2\frac{r}{k}(y,y),\quad (a',y)^2=\cos^2\frac{r'}{k}(y,y),\quad (a,y)\cos\frac{r'}{k}=(a',y)\cos\frac{r}{k},$$

$$(b,y)=0,\quad (b',y)=0,\quad (a,a)=(b,b)=1.$$

Hence, we have

$$\cos\theta=\frac{(a,a')\,(b,b')-(a,b')\,(a',b)-\cos\frac{r}{k}\cos\frac{r'}{k}\,(b,b')}{\sin\frac{r}{k}\,\sin\frac{r'}{k}}$$

$$=\frac{(p,p')\,(q,q')-(p,q')\,(p',q)-(q,q')\,\sqrt{(q,q)}\,\sqrt{(q',q')}}{\sqrt{(q,q)}\,\sqrt{(q',q')}\,\sqrt{(p,p)+(q,q)}\,\sqrt{(p',p')+(q',q')}}.$$

The condition of contact will be

$$\cos\theta=+1,$$

that is

$$(a,a')\,(b,b')-(a,b')\,(a',b)-\cos\frac{r'}{k}\cos\frac{r'}{k}(b,b')=\sin\frac{r}{k}\sin\frac{r'}{k}$$

and of orthogonal intersection

$$\sum\begin{vmatrix}a_i & a_j\\ b_i & b_j\end{vmatrix}\cdot\begin{vmatrix}a'_i & a'_j\\ b'_i & b'_j\end{vmatrix}=\cos\frac{r}{k}\cos\frac{r'}{k}\cos\psi$$

or

$$\frac{\sum\begin{vmatrix}p_i & p_j\\ q_i & q_j\end{vmatrix}\cdot\begin{vmatrix}p'_i & p'_j\\ q'_i & q'_j\end{vmatrix}}{\sqrt{\sum\begin{vmatrix}p_i & p_j\\ q_i & q_j\end{vmatrix}^2}\sqrt{\sum\begin{vmatrix}p'_i & p'_j\\ q'_i & q'_j\end{vmatrix}^2}}=(q,q'),\qquad (i,j=0,1,2,3),$$

where ψ is the angle between the planes of the circles.

If the given two circles are coplanar, then

$$\cos\theta=\frac{(a,a')-\cos\frac{r}{k}\cos\frac{r'}{k}}{\sin\frac{r}{k}\sin\frac{r}{k}}.$$ [1]

1 This is the formula which we introduced in §1. See also Coolidge, Non-Euclidean Geometry, p. 137.

Two circles cut orthogonally to each other, when one of the following conditions is satisfied:

(i) Two main spheres cut mutually orthogonally and one of the centers of the circles lies on the plane of the other.

(ii) Planes of the circles are mutually orthogonal and one of the centers of the circles lies on the plane of the other.

(iii) One of the main spheres is orthogonal to the other circle.

(iv) The space-axes are mutually orthogonal and the planes of the circles are orthogonal to each other.

If $\begin{pmatrix} p \\ q \end{pmatrix}$ and $\begin{pmatrix} p' \\ q' \end{pmatrix}$ are the coordinates of two circles Γ and Γ', the expression

$$\cos\left(\frac{\overline{aa'}}{k}\right)\cos\left(\frac{\overline{bb'}}{k}\right)-\cos\left(\frac{\overline{ab'}}{k}\right)\cos\left(\frac{\overline{ab}}{k}\right)-\cos\left(\frac{\overline{bb'}}{k}\right)\cos\frac{r}{k}\cos\frac{r'}{k}$$

shall be called *the mutual power of the circles* in analogy to the definition of that in the Spherical Trigonometry[1] and denote it by $P(\Gamma, \Gamma')$.

When the circles intersect, their mutual power is equal to the product of the sines of their radii divided by the measure of curvature and the cosine of their angle of intersection. When the circles do not intersect, their mutual power is not susceptible of this geometrical interpretation. In special case other interpretations can be found; thus if both circles be in the same plane, their mutual power is $\cos\left(\frac{\overline{aa'}}{k}\right)$ $-\cos\frac{r}{k}\cos\frac{r'}{k}$, and moreover in this case, if both circles be null their mutual power is $-2\sin^2\frac{1}{2}\left(\frac{\overline{aa'}}{k}\right)$.

§ 6

Involution and Bi-Involution[2]

Two circles shall be said to be in *involution* when each is orthogonal to a sphere through the other.

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1 See Todhunter and Leathem, Spherical Trigonometry.

2 This name is adopted in analogy to that of euclidean space given by Koenigs, See his remarkable article 'Contributions à la thêorie du cercle dans l'espace', Annales de la Faculté des Sciences de Toulouse, vol. ii, 1888.

Theorem. The necessary and sufficient condition that two circles should be in involution is that their mutual power is equal to zero.

Theorem. If the space-axes and the planes of two circles are mutually orthogonal, then they are in involution.

Theorem. When we construct two spheres through two non-coplaner circles, each center lies on the plane of the other, if the first sphere pass through the foci of the second circle, then the second sphere will also pass through the foci of the first, and they are in involution.

Theorem. If two circles intersect and are in involution, then they cut orthogonally to each other.

Next, we shall define two circles as being in *bi-involution* when every sphere through one is orthogonal to the other.

Theorem. The necessary and sufficient conditions that two non-coplanar and non-cospherical circles should be in bi-involution are that

$$(p, q')=0,\ (p', q)=0,\ (q, q')=0,$$

and

$$(p, p')+\sqrt{(q, q)}\sqrt{(q', q')}=0,$$

that is

(i) The centers of the circles lie on the intersection of the planes of the circles.

(ii) The planes of the circles are orthogonal,

(iii) Their main spheres cut orthogonally to each other.

Theorem. When two circles intersect and are in bi-involution, then they cut mutually orthogonally.

§ 7

Pseudo-Distances and Angles of two Non-Intersecting Circles.

Let $\begin{pmatrix} p \\ q \end{pmatrix}$, $\begin{pmatrix} p' \\ q' \end{pmatrix}$ be the coordinates of two non-coplanar and non-cospherical circles Γ and Γ', then the coordinates of the center of the common orthogonal sphere are

$$\lambda a_i=\frac{\partial}{\partial t_i}\begin{vmatrix} t & a\cos\frac{r'}{k}-a'\cos\frac{r}{k} & b & b' \end{vmatrix}$$

where λ is the propotional factor.

Consider the polar lines of (a), the center of the common orthogonal sphere, with respect to the given circles and denote them by l, l'. We shall mean the pseudo-distances of the two non-intersecting circles the distances of the lines l and l'.

As the polar plane of (a) with respect to one of the main spheres is given by the equation

$$(a, a)\,(a, x) - \cos^2\frac{r}{k}\,(a, x) = 0,$$

the axial coordinates of the polar lines l and l' are

$$l_{ij} = \begin{vmatrix} (a, a)\ a_i - \cos^2\frac{r}{k}\,a_i & (a, a)\ a_j - \cos^2\frac{r}{k}\,a_j \\ b_i & b_j \end{vmatrix}$$

and

$$l'_{ij} = \begin{vmatrix} (a', a)\ a'_i - \cos^2\frac{r'}{k}\,a_i & (a', a)\ a'_j - \cos^2\frac{r'}{k}\,a_i \\ b'_i & b'_j \end{vmatrix}.$$

Now, if we put

$$\mathfrak{X}_{ij} \equiv \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}, \quad \mathfrak{X}'_{ij} \equiv \begin{vmatrix} a'_i & a'_j \\ b'_i & b'_j \end{vmatrix}, \quad \mathfrak{Y}_{ij} \equiv \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}, \quad \mathfrak{Y}'_{ij} \equiv \begin{vmatrix} a_i & a_j \\ b'_i & b'_j \end{vmatrix},$$

$$(i, j = 0,\ 1,\ 2,\ 3),$$

then we get

$$l_{ij} = (a, a)\,\mathfrak{X}_i - \cos^2\frac{r}{k}\,\mathfrak{Y}_{ij},$$

$$l'_{ij} = (a', a)\,\mathfrak{X}'_{ij} - \cos^2\frac{r'}{k}\,\mathfrak{Y}'_{ij}.$$

As (a) are the coordinates of the center of the common orthogonal sphere, we have the following relations:

$$\frac{(a, a)}{\sqrt{(a, a)}} = \cos\frac{r}{k}\cos\frac{R}{k}, \quad \frac{(a', a)}{\sqrt{(a, a)}} = \cos\frac{r'}{k}\cos\frac{R}{k}$$

where R is the radius of the sphere. Hence, we have

$$\Sigma l_{ij}^2=(a,a)^2\ (\Sigma \mathcal{X}_{ij}^2)-2\ (a,a)\ (\Sigma \mathcal{X}_{ij}\mathcal{Y}_{ij})\ \cos^2\frac{r}{k}+(\Sigma \mathcal{Y}_{ij}^2)\ \cos^4\frac{r}{k}$$

$$=(a,a)\cos^2\frac{r}{k}\left\{\cos^2\frac{R}{k}\sin^2\frac{r}{k}+\sin^2\frac{R}{k}\cos^2\frac{r}{k}\right\},$$

$$\Sigma l'^2_{ij}=(a,a)\cos^2\frac{r'}{k}\left\{\cos^2\frac{R}{k}\sin^2\frac{r'}{k}+\sin^2\frac{R}{k}\cos^2\frac{r'}{k}\right\},$$

$$\Sigma l'_{ij}l'_{ij}=(a,a)\cos\frac{r}{k}\cos\frac{r'}{k}\left\{\cos^2\frac{R}{k}P(\Gamma,\Gamma')+\cos\psi\cos\frac{r}{k}\cos\frac{r'}{k}\sin^2\frac{R}{k}\right\},$$

where $P(\Gamma, \Gamma')$ is the mutual power of the two circles and ψ is the angle of the planes of the circles.

And

$$(l\,|\,l')\equiv\Sigma l_{ij}\ l'_{kl}\qquad(i,\ j,\ k,\ l=0,\ 1,\ 2,\ 3)$$

$$=(a,a)\ (a',a)\ (\mathcal{X}|\mathcal{X}')-(a',a)\ \cos^2\frac{r}{k}(\mathcal{X}'|\mathcal{Y})-(a,a)\cos^2\frac{r'}{k}(\mathcal{X}\,|\,\mathcal{Y}')$$

$$+\cos^2\frac{r}{k}\cos^2\frac{r'}{k}(\mathcal{Y}\,|\,\mathcal{Y}')$$

$$=(a,a)\sqrt{(a,a)}\ \cos\frac{r}{k}\cos\frac{r'}{k}\cos^3\frac{R}{k}-(a,a)\sqrt{(a,a)}\ \cos\frac{r}{k}\cos\frac{r'}{k}\cos\frac{R}{k}$$

$$=-\ (a,a)\sqrt{(a,a)}\cos\frac{r}{k}\cos\frac{r'}{k}\cos\frac{R}{k}\sin^2\frac{R}{k}$$

$$=-(aa)\begin{vmatrix} a_0 & a_1 & a_2 & a_3\\ b_0 & b_1 & b_2 & b_3\\ a'_0 & a'_1 & a'_2 & a'_3\\ b'_0 & b'_1 & b'_2 & b'_3\end{vmatrix}\cos\frac{r}{k}\cos\frac{r'}{k}\sin^2\frac{R}{k}$$

Let d_1 and d_2 be the distances of the lines l and l', then the square of their moment[1] will be

$$\sin^2\frac{d_1}{k}\sin^2\frac{d_2}{k}=\frac{(l\,|\,l')^2}{(\Sigma l_{ij}^2)\,.\,(\Sigma l'^2_{ij})}\ ^{(2)}$$

1 See D'Ovidio, 'Studio sulla geometria proiettiva,' Annali di mathematica, vi, 1873.

2 See Coolidge, Non-Euclidean Geometry, p. 112.

$$= \frac{\left| a \;\; b \;\; a' \;\; b' \right|^2 \tan^4 \frac{R}{k}}{\left(\sin^2 \frac{r}{k} + \cos^2 \frac{r}{k} \tan^2 \frac{R}{k} \right) \left(\sin^2 \frac{r'}{k} + \cos^2 \frac{r'}{k} \tan^2 \frac{R}{k} \right)}$$

and

$$\sin^2 \frac{d_1}{k} + \sin^2 \frac{d_2}{k}$$

$$= 1 + \frac{\left| a\,b\,a'b' \right|^2 \tan^4 \frac{R}{k} - \left\{ P(\Gamma, \Gamma') + \cos\psi \cos \frac{r}{k} \cos \frac{r'}{k} \tan^2 \frac{R}{k} \right\}^2}{\left(\sin^2 \frac{r}{k} + \cos^2 \frac{r}{k} \tan^2 \frac{R}{k} \right) \left(\sin^2 \frac{r'}{k} + \cos^2 \frac{r'}{k} \tan^2 \frac{R}{k} \right)}. \quad (1)$$

From these equations, we can find the pseudo-distances of the circles.

And the square of the commoment[2] of the lines l and l' is

$$\cos^2 \frac{d_1}{k} \cos^2 \frac{d_2}{k} = \frac{(\Sigma l_{ij}\, l'_{ij})^2}{(\Sigma l_{ij}^2)\,(\Sigma l'^2_{ij})} \quad (3)$$

$$= \frac{\left\{ P(\Gamma, \Gamma') + \cos\psi \cos \frac{r}{k} \cos \frac{r'}{k} \tan^2 \frac{R}{k} \right\}^2}{\left(\sin^2 \frac{r}{k} + \cos^2 \frac{r}{k} \tan^2 \frac{R}{k} \right) \left(\sin^2 \frac{r'}{k} + \cos^2 \frac{r'}{k} \tan^2 \frac{R}{k} \right)}.$$

We shall call the momment (commoment) of the lines l and l' the pseudo-moment (pseudo-commoment) of the circles. When the circles intersect or their space-axes cut each other, their pseudo-moment vanishes, and when their mutual power is equal to zero and the planes of the circles cut orthogonally then their pseudo-comment vanishes.

We shall call the angles of the lines l and l' the pseudo-angles of the circles.

Let θ_1 and θ_2 be the angles of the lines l and l', i.e. the pseudo-angles of the circles, then

$$\sin^2\theta_1 \;\; \sin^2\theta_2 = \frac{\left| a \;\; b \;\; a' \;\; b' \right|^2 \tan^4 \frac{R}{k}}{\left(\sin^2 \frac{r}{k} + \cos^2 \frac{r}{k} \tan^2 \frac{R}{k} \right) \left(\sin^2 \frac{r'}{k} + \cos^2 \frac{r'}{k} \tan^2 \frac{R}{k} \right)},$$

1, 3 See Coolidge, Non-Euclidean Geometry, p. 112.

2 See D'Ovidio, 'Studio sulla geometria proiettiva,' Annali di mathematica, vi, 1873.

and

$$\cos^2\theta_1\,\cos^2\theta_2 = \frac{\left\{P(\Gamma,\Gamma')+\cos\psi\cos\frac{r}{k}\cos\frac{r'}{k}\tan^2\frac{R}{k}\right\}^2}{\left(\sin^2\frac{r}{k}+\cos^2\frac{r}{k}\tan^2\frac{R}{k}\right)\left(\sin^2\frac{r'}{k}+\cos^2\frac{r'}{k}\tan^2\frac{R}{k}\right)}.$$

When the circles intersect, then

$$\cos^2\theta = \frac{\{P(\Gamma,\Gamma')\}^2}{\sin^2\frac{r}{k}\,\sin^2\frac{r'}{k}}.$$

This is the formula which we reduced in § 5.

Two circles shall be said to be pseudo-paratactic[1] when their pseudo-distances and pseudo-angles are all congruent.

CHAPTER II.

§ 8

Other Coordinates of Circle.

Next, we put

$$i\cos\frac{r}{k}b_0 \equiv \mathfrak{X}_0,\ i\cos\frac{r}{k}b_1 \equiv \mathfrak{X}_1,\ i\cos\frac{r}{k}b_2 \equiv \mathfrak{X}_2,\ i\cos\frac{r}{k}b_3 \equiv \mathfrak{X}_3,$$

$$\begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix} \equiv \mathfrak{X}_{01},\ \begin{vmatrix} a_0 & a_2 \\ b_0 & b_2 \end{vmatrix} \equiv \mathfrak{X}_{02},\ \begin{vmatrix} a_0 & a_3 \\ b_0 & b_3 \end{vmatrix} \equiv \mathfrak{X}_{03},$$

$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \equiv \mathfrak{X}_{23},\ \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} \equiv \mathfrak{X}_{31},\ \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \equiv \mathfrak{X}_{12},\ (i=\sqrt{-1}),$$

where (a_0, a_1, a_2, a_3), (b_0, b_1, b_2, b_3) and r have the same meaning as in § 1.

We can take $(\mathfrak{X}_0, \mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3, \mathfrak{X}_{01}, \mathfrak{X}_{02}, \mathfrak{X}_{03}, \mathfrak{X}_{31}, \mathfrak{X}_{12}) \equiv (\mathfrak{X})$ as the coordinates of a circle. This is analogous to the coordinates of 'the Soma of the second sort' given by Study.[2] So we shall call

1 This word is adopted in analogy to the word 'paratactic.' See Study, 'Zur Nicht-Euclidischen und Linien-Geometrie,' Jahers-bericht der deutschen Mathematikervereinigung, xi, 1902.

2 See Study, Geomertie der Dynamen.

this coordinates *Study's coordinates of the circle.* It is evident that there exists the following relation:

$$\mathfrak{X}_{01}\mathfrak{X}_{23}+\mathfrak{X}_{02}\mathfrak{X}_{31}+\mathfrak{X}_{03}\mathfrak{X}_{12}=0.$$

When we have two circles Γ and Γ' with Study's coordinates $(\mathfrak{X})$ and $(\mathfrak{X}')$, then

$$P(\Gamma,\Gamma')=(\Sigma\mathfrak{X}_i\mathfrak{X}'_i)+\Sigma\mathfrak{X}_{ij}\mathfrak{X}'_{ij},\ (i,j=0,\ 1,\ 2,\ 3).$$

If we adopt the following notation

$$[\mathfrak{X},\mathfrak{X}]\equiv\Sigma\mathfrak{X}_i^2+\Sigma\mathfrak{X}_{ij}^2,$$

$$[\mathfrak{X},\mathfrak{X}']\equiv\Sigma\mathfrak{X}_i\mathfrak{X}'_i+\Sigma\mathfrak{X}_{ij}\mathfrak{X}'_{ij},\ (i,j=0,\ 1,\ 2,\ 3),$$

then

$$P(\Gamma,\Gamma')=[\mathfrak{X},\mathfrak{X}'].$$

If

$$\frac{\mathfrak{X}_i}{\sqrt{\Sigma\mathfrak{X}_i^2}}=\frac{\mathfrak{X}'_i}{\sqrt{\Sigma\mathfrak{X}_i'^2}},\ (i=0,\ 1,\ 2,\ 3),$$

then they are coplanar.

If

$$\mathfrak{X}_0^2+\mathfrak{X}_1^2+\mathfrak{X}_2^2+\mathfrak{X}_3^2=\mathfrak{X}_0'^2+\mathfrak{X}_1'^2+\mathfrak{X}_2'^2+\mathfrak{X}_3'^2,$$

then they are equiradii.

If

$$\mathfrak{X}_{ij}=\mathfrak{X}'_{ij},\ (i,j=0,\ 1,\ 2,\ 3),$$

then they have the same space-axis.

If

$$(\mathfrak{X}_i,\mathfrak{X}'_i)\equiv\mathfrak{X}_0\mathfrak{X}'_0+\mathfrak{X}_1\mathfrak{X}'_1+\mathfrak{X}_2\mathfrak{X}'_2+\mathfrak{X}_3\mathfrak{X}'_3=0,$$

then the planes of the circles cut mutually orthogonally.

If

$$(\mathfrak{X}_{ij},\mathfrak{X}'_{ij})=\Sigma\mathfrak{X}_{ij}\mathfrak{X}'_{ij}=0,\ (i,j=0,\ 1,\ 2,\ 3)$$

then their space-axes are mutually absolute polars.

If

$$(\mathfrak{X}_{ij}\mid\mathfrak{X}'_{ij})=\Sigma\mathfrak{X}_{ij}\mathfrak{X}'_{kl}=0,(i,\ j,\ k,\ l,=0,\ 1,\ 2,\ 3,$$

then their space-axes cut each other.

If

$$(\mathfrak{X}_{ij}\mid\mathfrak{X}'_{ij})=(\mathfrak{X}_{ij},\mathfrak{X}'_{ij})=0,$$

then their space-axes cut orthogonally to each other.

If

$$(\mathfrak{X}_{ij} \mid \mathfrak{X}_{ij}) = [(\mathfrak{X}_{ij}, \mathfrak{X}_{ij})\ (\mathfrak{X}'_{ij}, \mathfrak{X}'_{ij}) - (\mathfrak{X}_{ij}, \mathfrak{X}'_{ij})^2] = 0,$$

then their space-axes are either parallel or pseudo-parallel.

If they intersect each other, their angles of intersection is given by the formula:

$$\cos\theta = \frac{[\mathfrak{X}, \mathfrak{X}']}{\sqrt{[\mathfrak{X}, \mathfrak{X}]}\sqrt{[\mathfrak{X}', \mathfrak{X}']}}.$$

If they cut orthogonally or their mutual power is equal to zero, then

$$[\mathfrak{X}, \mathfrak{X}'] = 0$$

or

$$(\mathfrak{X}_i, \mathfrak{X}'_i) + (\mathfrak{X}_{ij}, \mathfrak{X}'_{ij}) = 0,\ (i,j = 0,\ 1,\ 2,\ 3),$$

hence

$$\frac{(\mathfrak{X}_{ij}, \mathfrak{X}'_{ij})^2 - (\mathfrak{X}_i, \mathfrak{X}'_i)^2}{(\mathfrak{X}_{ij}, \mathfrak{X}_{ij})\ (\mathfrak{X}'_{ij}, \mathfrak{X}'_{ij})} = 0.$$

Therefore, the commoment of the space-axes of the circles is equal to

$$\cos\frac{r}{k}\cos\frac{r'}{k}\cos\psi,$$

where ψ is the angle between the planes of the circles.

Let d_1 and d_2 be the distances between their space-axes and θ_1 and θ_2 their angles, then

$$\cos^2\frac{d_1}{k}\cos^2\frac{d_2}{k} = \cos^2\frac{r}{k}\cos^2\frac{r'}{k}\cos^2\psi,$$

and

$$\cos^2\theta_1\cos^2\theta_2 = \cos^2\frac{r}{k}\cos^2\frac{r'}{k}\cos^2\psi.$$

As a special case, if the space-axes be paratactic and equiradii, then

$$d_1 = d_2 = d,\ r = r'$$

and

$$\cos^4\frac{d}{k} = \cos^4\frac{r}{k}\cos^2\psi.$$

Similarly, if their polar circles cut mutually orthogonally, then

$$\cos^2\frac{d_1}{k}\cos^2\frac{d_2}{k} = \sin^2\frac{r}{k}\sin^2\frac{r'}{k}\cos^2\psi.$$

If the circles are concentric, then

$$(\mathfrak{X}_{ij}, \mathfrak{X}'_{ij}) = \cos\phi.$$

The condition that they should be in involution is that

$$[\mathfrak{X}, \mathfrak{X}'] = 0,$$

and that of bi-involution is that

$$(\mathfrak{X}_i, \mathfrak{X}'_i) = 0,$$

$$\Sigma \mathfrak{X}_{ij} \begin{vmatrix} \mathfrak{X}_i & \mathfrak{X}_j \\ \mathfrak{X}'_i & \mathfrak{X}'_j \end{vmatrix} = \Sigma \mathfrak{X}'_{ij} \begin{vmatrix} \mathfrak{X}_i & \mathfrak{X}_j \\ \mathfrak{X}'_i & \mathfrak{X}'_j \end{vmatrix} = 0$$

and

$$(\mathfrak{X}_{ij} \mid \mathfrak{X}'_{ij})^2 + (\mathfrak{X}_i, \mathfrak{X}_i)\,(\mathfrak{X}'_i, \mathfrak{X}'_i) = (\mathfrak{X}_{ij}, \mathfrak{X}_{ij})\,(\mathfrak{X}'_{ij}, \mathfrak{X}'_{ij}).$$

In this case, the square of the moment of their space-axes is equal to

$$\sin^2\frac{d_1}{k}\sin^2\frac{d_2}{k} = 1 - \cos^2\frac{r}{k}\cos^2\frac{r'}{k} = \sin^2\left(\frac{\overline{aa'}}{k}\right),$$

i.e. when they are in bi-involution, the square of the moment of their space-axes is equal to the square of the sine of the distance of their centers divided by the measure of curvature.

If we take two systems of seven circles with Study's coordinates

$$(\mathfrak{X}^{(1)}), (\mathfrak{X}^{(2)}), \ldots\ldots, (\mathfrak{X}^{(7)}) \text{ and } (\mathfrak{Y}^{(1)}), (\mathfrak{Y}^{(2)}), \ldots\ldots, (\mathfrak{Y}^{(7)}),$$

and form the product of the two vanishing determinants,

$$\begin{vmatrix} \mathfrak{X}^{(1)}_{01} & \mathfrak{X}^{(1)}_{02} & \mathfrak{X}^{(1)}_{03} & \mathfrak{X}^{(1)}_{23} & \mathfrak{X}^{(1)}_{31} & \mathfrak{X}^{(1)}_{12} & 0 \\ \mathfrak{X}^{(2)}_{01} & \mathfrak{X}^{(2)}_{02} & \mathfrak{X}^{(2)}_{03} & \mathfrak{X}^{(2)}_{23} & \mathfrak{X}^{(2)}_{31} & \mathfrak{X}^{(2)}_{12} & 0 \\ \cdots\cdots & & & & & & \\ \cdots\cdots & & & & & & \\ \mathfrak{X}^{(7)}_{01} & \mathfrak{X}^{(7)}_{02} & \mathfrak{X}^{(7)}_{03} & \mathfrak{X}^{(7)}_{23} & \mathfrak{X}^{(7)}_{31} & \mathfrak{X}^{(7)}_{12} & 0 \end{vmatrix}$$

and

$$\begin{vmatrix} \mathfrak{Y}^{(1)}_{01} & \mathfrak{Y}^{(1)}_{02} & \mathfrak{Y}^{(1)}_{03} & \mathfrak{Y}^{(1)}_{23} & \mathfrak{Y}^{(1)}_{31} & \mathfrak{Y}^{(1)}_{12} & 0 \\ \mathfrak{Y}^{(2)}_{01} & \mathfrak{Y}^{(2)}_{02} & \mathfrak{Y}^{(2)}_{03} & \mathfrak{Y}^{(2)}_{23} & \mathfrak{Y}^{(2)}_{31} & \mathfrak{Y}^{(2)}_{12} & 0 \\ \cdots\cdots & & & & & & \\ \cdots\cdots & & & & & & \\ \mathfrak{Y}^{(7)}_{01} & \mathfrak{Y}^{(7)}_{02} & \mathfrak{Y}^{(7)}_{03} & \mathfrak{Y}^{(7)}_{23} & \mathfrak{Y}^{(7)}_{31} & \mathfrak{Y}^{(7)}_{12} & 0 \end{vmatrix}$$

then we have the following identity:

$$\begin{vmatrix} \cos\frac{d_{11'}}{k} \cos\frac{d''_{11'}}{k} & \cos\frac{d_{12'}}{k} \cos\frac{d''_{12'}}{k} & \cdots\cdots & \cos\frac{d_{17'}}{k} \cos\frac{d''_{17'}}{k} \\ \cos\frac{d_{21'}}{k} \cos\frac{d''_{21'}}{k} & \cos\frac{d_{22'}}{k} \cos\frac{d''_{22'}}{k} & \cdots\cdots & \cos\frac{d_{27'}}{k} \cos\frac{d''_{27'}}{k} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cos\frac{d_{71'}}{k} \cos\frac{d''_{71'}}{k} & \cos\frac{d_{72'}}{k} \cos\frac{d''_{72'}}{k} & \cdots\cdots & \cos\frac{d_{77'}}{k} \cos\frac{d''_{77'}}{k} \end{vmatrix} = 0,$$

where $d_{ij'}$ and $d''_{ij'}$ are the distances of the space-axes of the circles with coordinates $(\mathfrak{X}^{(i)})$ and $(\mathfrak{Y}^{(j)})$.

If two sets of circles coincide respectively, then

$$\begin{vmatrix} \text{I} & \cos\frac{d_{12}}{k} \cos\frac{d''_{12}}{k} & \cdots\cdots & \cos\frac{d_{17}}{k} \cos\frac{d''_{17}}{k} \\ \cos\frac{d_{21}}{k} \cos\frac{d''_{21}}{k} & \text{I} & \cdots\cdots & \cos\frac{d_{27}}{k} \cos\frac{d''_{27}}{k} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cos\frac{d_{71}}{k} \cos\frac{d''_{71}}{k} & \cos\frac{d_{72}}{k} \cos\frac{d''_{72}}{k} & \cdots\cdots & \text{I} \end{vmatrix} = 0,$$

where d_{ij} and d''_{ij} are the distances of the space-axes of the circles with coordinates $(\mathfrak{X}^{(i)})$ and $(\mathfrak{X}^{(j)})$.

Moreover, if each of these seven circles cut mutually orthogoally to the others, or if they are in involution to one another, then

$$\begin{vmatrix} \text{I} & \cos\frac{r_1}{k}\cos\frac{r_2}{k}\cos\psi_{12} & \cdots\cdots & \cos\frac{r_1}{k}\cos\frac{r_7}{k}\cos\psi_{17} \\ \cos\frac{r_2}{k}\cos\frac{r_1}{k}\cos\psi_{21} & \text{I} & \cdots\cdots & \cos\frac{r_2}{k}\cos\frac{r_7}{k}\cos\psi_{27} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cos\frac{r_7}{k}\cos\frac{r_1}{k}\cos\psi_{71} & \cos\frac{r_7}{k}\cos\frac{r_2}{k}\cos\psi_{72} & \cdots\cdots & \text{I} \end{vmatrix} = 0,$$

that is

$$\cos\frac{r_1}{k}\cos\frac{r_2}{k} \cdots\cdots \frac{r_7}{k} = 0,$$

or

$$\begin{vmatrix} \frac{1}{\cos^2\frac{r_1}{k}} & \cos\psi_{12} & \cdots\cdots & \cos\psi_{17} \\ \cos\psi_{21} & \frac{1}{\cos^2\frac{r_2}{k}} & \cdots\cdots & \cos\psi_{27} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cos\psi_{71} & \cos\psi_{72} & \cdots\cdots & \frac{1}{\cos^2\frac{r_7}{k}} \end{vmatrix} = 0,$$

where ψ_{ij} is the angle between the planes of the circles with coordinates $(\mathfrak{X}^{(i)})$ and $(\mathfrak{X}^{(j)})$.

Similarly, we get the following identity among the seven circles

$$\begin{vmatrix} 0 & \sin\frac{d_{12}}{k}\sin\frac{d'_{12}}{k} & \cdots\cdots & \sin\frac{d_{17}}{k}\sin\frac{d'_{11}}{k} \\ \sin\frac{d_{21}}{k}\sin\frac{d'_{21}}{k} & 0 & \cdots\cdots & \sin\frac{a_{27}}{k}\sin\frac{d'_{27}}{k} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \sin\frac{d_{71}}{k}\sin\frac{d'_{71}}{k} & \sin\frac{d_{72}}{k}\sin\frac{d'_{72}}{k} & \cdots\cdots & 0 \end{vmatrix} = 0.$$

If they are in bi-involution to one another, then

$$\begin{vmatrix} 0 & \sin\left(\frac{\overline{a_1a_2}}{k}\right) & \cdots\cdots & \sin\left(\frac{\overline{a_1a_7}}{k}\right) \\ \sin\left(\frac{\overline{a_2a_1}}{k}\right) & 0 & \cdots\cdots & \sin\left(\frac{\overline{a_2a_7}}{k}\right) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \sin\left(\frac{\overline{a_7a_1}}{k}\right) & \sin\left(\frac{\overline{a_7a_2}}{k}\right) & \cdots\cdots & 0 \end{vmatrix} = 0,$$

where $\sin\left(\frac{\overline{a_ia_j}}{k}\right)$ is the sine of the distance between the centers of the circles with coordinates $(\mathfrak{X}^{(i)}),(\mathfrak{X}^{(j)})$ divided by the measure of curvature.

If $(\overline{\mathfrak{X}})$ be the coordinates of the polar circle of a given circle with coordinates $(\mathfrak{X})$, then

$$[\mathfrak{X}, \mathfrak{X}]+[\overline{\mathfrak{X}}, \overline{\mathfrak{X}}]=1.$$

§ 9

Cospherical Circles.

If a circle with Study's coordinates $(\mathfrak{X})$ lies on a sphere

$$(a, x)^2 = \cos^2\frac{R}{k}(xx), \quad (a, a) = 1,$$

then the coordinates of the center of the circle is

$$a_i = \frac{a_i - (b, a)\, b_i}{\sqrt{1-(b, a)^2}},$$

and

$$\cos\frac{r}{k} = \frac{1}{\sqrt{1-(b, a)^2}}\cos\frac{R}{k}, \quad (i=0, 1, 2, 3).$$

Now, we put

$$\mathfrak{X}_0, \mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3, \mathfrak{X}_{01}, \mathfrak{X}_{02}, \mathfrak{X}_{03}, \mathfrak{X}_{23}, \mathfrak{X}_{31}, \mathfrak{X}_{12}$$

$$\equiv i\cos\frac{R}{k}b_0, i\cos\frac{R}{k}b_1, i\cos\frac{R}{k}b_2, i\cos\frac{R}{k}b_3,$$

$$\begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix}, \begin{vmatrix} a_0 & a_2 \\ b_0 & b_2 \end{vmatrix}, \begin{vmatrix} a_0 & a_3 \\ b_0 & b_3 \end{vmatrix}, \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

If two circles Γ and Γ' lie on the same sphere

$$(a, x)^2 = \cos^2\frac{R}{k}(x, x), \quad (a, a) = 1,$$

then the angles of the intersection is given by the following formula:

$$\cos\theta = \frac{(a, a')(b, b') - (a, b')(a, b') - (b, b')\cos\frac{r}{k}\cos\frac{r'}{k}}{\sin\frac{r}{k}\sin\frac{r'}{k}}$$

$$= \frac{(b,b') - (a,b)(a,b') - (b,b')\cos^2\frac{R}{k}}{\sqrt{\sin^2\frac{R}{k} - (b, a)^2}\sqrt{\sin^2\frac{R}{k} - (b', a)^2}} = \frac{[\mathfrak{X}, \mathfrak{X}']}{\sqrt{[\mathfrak{X}, \mathfrak{X}]}\sqrt{[\mathfrak{X}', \mathfrak{X}']}}.$$

And their mutual power is

$$P(\Gamma,\Gamma') = \frac{(b,b')-(a,b)(a,b')-(b,b')\cos^2\frac{R}{k}}{\sqrt{1-(b,a)^2}\sqrt{1-(b',a)^2}}$$

$$= \frac{[\mathfrak{X},\mathfrak{X}']}{\sqrt{(\mathfrak{X}_{ij},\mathfrak{X}_{ij})}\sqrt{(\mathfrak{X}'_{ij},\mathfrak{X}'_{ij})}}.$$

If the planes of the circles pass through the center of the sphere, then they are equiradii and

$$\cos\theta = \cos\psi$$

where ψ is the angle between the planes of the circles.

If

$$[\mathfrak{X},\mathfrak{X}] = [\mathfrak{X}',\mathfrak{X}']$$

then they are equiradii.

If

$$\Sigma\mathfrak{X}_{ij}\begin{vmatrix}\mathfrak{X}_i & \mathfrak{X}_j\\ \mathfrak{X}'_i & \mathfrak{X}'_j\end{vmatrix} = 0,$$

or

$$\Sigma\mathfrak{X}'_{ij}\begin{vmatrix}\mathfrak{X}_i & \mathfrak{X}_j\\ \mathfrak{X}'_i & \mathfrak{X}'_j\end{vmatrix} = 0,$$

then one of the centers of the circles is on the plane of the othere.

If

$$\Sigma\mathfrak{X}_{ij}\begin{vmatrix}\mathfrak{X}_i & \mathfrak{X}_j\\ \mathfrak{X}'_i & \mathfrak{X}'_j\end{vmatrix} = \Sigma\mathfrak{X}'_{ij}\begin{vmatrix}\mathfrak{X}_i & \mathfrak{X}_j\\ \mathfrak{X}'_i & \mathfrak{X}'_j\end{vmatrix} = 0,$$

then their centers coincide with that of the sphere (they are concentric), or they are coplanar.

§ 10

On the Mutual Power.

Let there be two systems of circles, each consisting of eleven circles, namely $\Gamma_1, \Gamma_2, \Gamma_3, \ldots\ldots, \Gamma_{11}$, and $\Pi_1, \Pi_2, \Pi_3, \ldots\ldots, \Pi_{11}$, whose Study's coordinates are $(\mathfrak{X}^{(1)}), (\mathfrak{X}^{(2)}), \ldots, (\mathfrak{X}^{(11)})$ and $(\mathfrak{Y}^{(1)}), (\mathfrak{Y}^{(2)}), \ldots, (\mathfrak{Y}^{(11)})$, respectively. Form the product of two vanishing determinants

$$\begin{vmatrix} \mathcal{X}_0^{(1)} & \mathcal{X}_1^{(1)} & \mathcal{X}_2^{(1)} & \mathcal{X}_3^{(1)} & \mathcal{X}_{01}^{(1)} & \mathcal{X}_{02}^{(1)} & \mathcal{X}_{03}^{(1)} & \mathcal{X}_{23}^{(1)} & \mathcal{X}_{31}^{(1)} & \mathcal{X}_{12}^{(1)} & 0 \\ \mathcal{X}_0^{(2)} & \mathcal{X}_1^{(2)} & \mathcal{X}_2^{(2)} & \mathcal{X}_3^{(2)} & \mathcal{X}_{01}^{(2)} & \mathcal{X}_{02}^{(2)} & \mathcal{X}_{03}^{(2)} & \mathcal{X}_{23}^{(2)} & \mathcal{X}_{31}^{(2)} & \mathcal{X}_{12}^{(2)} & 0 \\ \cdots & & & & & & & & & & \\ \cdots & & & & & & & & & & \\ \mathcal{X}_0^{(11)} & \mathcal{X}_1^{(11)} & \mathcal{X}_2^{(11)} & \mathcal{X}_3^{(11)} & \mathcal{X}_{01}^{(11)} & \mathcal{X}_{02}^{(11)} & \mathcal{X}_{03}^{(11)} & \mathcal{X}_{23}^{(11)} & \mathcal{X}_{31}^{(11)} & \mathcal{X}_{12}^{(11)} & 0 \end{vmatrix}$$

and

$$\begin{vmatrix} \mathcal{Y}_0^{(1)} & \mathcal{Y}_1^{(1)} & \mathcal{Y}_2^{(1)} & \mathcal{Y}_3^{(1)} & \mathcal{Y}_{01}^{(1)} & \mathcal{Y}_{02}^{(1)} & \mathcal{Y}_{03}^{(1)} & \mathcal{Y}_{23}^{(1)} & \mathcal{Y}_{31}^{(1)} & \mathcal{Y}_{12}^{(1)} & 0 \\ \mathcal{Y}_0^{(2)} & \mathcal{Y}_1^{(2)} & \mathcal{Y}_2^{(2)} & \mathcal{Y}_3^{(2)} & \mathcal{Y}_{01}^{(2)} & \mathcal{Y}_{02}^{(2)} & \mathcal{Y}_{03}^{(2)} & \mathcal{Y}_{23}^{(2)} & \mathcal{Y}_{31}^{(2)} & \mathcal{Y}_{12}^{(2)} & 0 \\ \cdots & & & & & & & & & & \\ \cdots & & & & & & & & & & \\ \mathcal{Y}_0^{(11)} & \mathcal{Y}_1^{(11)} & \mathcal{Y}_2^{(11)} & \mathcal{Y}_3^{(11)} & \mathcal{Y}_{01}^{(11)} & \mathcal{Y}_{02}^{(11)} & \mathcal{Y}_{03}^{(11)} & \mathcal{Y}_{23}^{(11)} & \mathcal{Y}_{31}^{(11)} & \mathcal{Y}_{12}^{(11)} & 0 \end{vmatrix}$$

and we get immediately following result analogous with Frobenius and Darboux's identity.[1]

$$\begin{vmatrix} P(\Gamma_1, \Pi_1) & P(\Gamma_1, \Pi_2) & \cdots & P(\Gamma_1, \Pi_{11}) \\ P(\Gamma_2, \Pi_1) & P(\Gamma_2, \Pi_2) & \cdots & P(\Gamma_2, \Pi_{11}) \\ \cdots & & & \\ \cdots & & & \\ P(\Gamma_{11}, \Pi_1) & P(\Gamma_{11}, \Pi_2) & \cdots & P(\Gamma_{11}, \Pi_{11}) \end{vmatrix} = 0. \qquad (1)$$

If each circle of one system intersect all the circles of the other system, the angle of intersection of the circles Γ_m anc Π_n being represented by (m,n'), we derive the relation:

$$\begin{vmatrix} cos\ (1, 1') & cos\ (1, 2') & \cdots & cos\ (1, 11') \\ cos\ (2, 1') & cos\ (2, 2') & \cdots & cos\ (2, 11') \\ \cdots & & & \\ \cdots & & & \\ cos\ (11, 1') & cos\ (11, 2') & \cdots & cos\ (11, 11') \end{vmatrix} = 0. \qquad (2)$$

If the first ten circles of the first system are cut orthogonally by a same circle K′, and if we take K′ to be the eleventh circle of the second system, then the first ten elements of the last column vanish in this formula. Hence we get the following relation between the angles in which a system of ten circles, possessed of a common orthogonal circle, are cut by any other ten circles:

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1 See G. Frobenius, 'Anwendungen der Determinanten Theorie auf die Geometrie der Masses,' Crelle's Journal, vol. LXXIX, 1875, p. 187. See also Cayley, Cambridge Mathematical Journal, vol. II, 1841, p. 267, or Collected Works, vol. I, p. 1.

See Darboux, 'Groupes de points de cercles et de sphares,' Annales de l'Ecole Normale, Series, 2, vol. i, 1872.

$$\begin{vmatrix} \cos(1,1') & \cos(1,2') & \cdots & \cos(1,10') \\ \cos(2,1') & \cos(2,2') & \cdots & \cos(1,10') \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cos(10,1') & \cos(10,2') & \cdots & \cos(10,10') \end{vmatrix} = 0. \qquad (3)$$

If K be a circle which cuts $\Pi_1, \Pi_2, \ldots\ldots, \Pi_9$ orthogonally, and if K′ cut K orthogonally, K may be taken as the tenth circle of the first system. Then we get the following relation between the angles of intersection of the two systems of nine circles, whose respective orthogonal circles cut at right angles:

$$\begin{vmatrix} \cos(1,1') & \cos(1,2') & \cdots & \cos(1,9') \\ \cos(2,1') & \cos(2,2') & \cdots & \cos(2,9') \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cos(9,1') & \cos(9,2') & \cdots & \cos(9,9') \end{vmatrix} = 0. \qquad (4)$$

If the second system of nine circles coincide with the first respectively, then

$$\begin{vmatrix} 1 & \cos(1,2) & \cdots & \cos(1,9) \\ \cos(2,1) & 1 & \cdots & \cos(2,9) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cos(9,1) & \cos(9,2) & \cdots & 1 \end{vmatrix} = 0. \qquad (5)$$

In the relation (1), let the two sets of circles coincide, then it appears that any eleven circles satisfy the relation:

$$\begin{vmatrix} \sin^2\frac{r_1}{k} & P(\Gamma_1,\Gamma_2) & \cdots & P(\Gamma_1,\Gamma_{11}) \\ P(\Gamma_2,\Gamma_1) & \sin^2\frac{r_2}{k} & \cdots & P(\Gamma_2,\Gamma_{11}) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ P(\Gamma_{11},\Gamma_1) & P(\Gamma_{11},\Gamma_2) & \cdots & \sin^2\frac{r_{11}}{k} \end{vmatrix} = 0. \qquad (6)$$

Thus the condition that the first ten circles are cut orthogonally by the eleventh is that

$$\begin{vmatrix} \sin^2\frac{r_1}{k} & P(\Gamma_1, \Gamma_2) & \dots & P(\Gamma_1, \Gamma_{10}) \\ P(\Gamma_2, \Gamma_1) & \sin^2\frac{r_2}{k} & \dots & P(\Gamma_2, \Gamma_{10}) \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ P(\Gamma_{10}, \Gamma_1) & P(\Gamma_2, \Gamma_2) & \dots & \sin^2\frac{r_{10}}{k} \end{vmatrix} = 0. \qquad (7)$$

And the condition that the first ten circles touch the eleventh is that (when they intersect one another)

$$\begin{vmatrix} 1 & \cos(1, 2) & \dots & \cos(1, 10) & 1 \\ \cos(2, 1) & 1 & \dots & \cos(2, 10) & 1 \\ \dots & \dots & \dots & \dots & \dots \\ \cos(10, 1) & \cos(10, 2) & \dots & 1 & 1 \\ 1 & 1 & \dots & 1 & 1 \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} 0 & \sin^2\frac{(1, 2)}{2} & \dots & \sin^2\frac{(1, 10)}{2} \\ \sin^2\frac{(2, 1)}{2} & 0 & \dots & \sin^2\frac{(2, 10)}{2} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \sin^2\frac{(10, 1)}{2} & \sin^2\frac{(10, 2)}{2} & \dots & 0 \end{vmatrix} = 0. \qquad (8)$$

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